Mathematics

Geometry

DETROIT
PUBLIC SCHOOLS
COMMUNITY DISTRICT

Letter to Families from the DPSCD Office of Mathematics

Dear DPSCD Families,

The Office of Mathematics is partnering with families to support Distance Learning while students are home. We empower you to utilize the resources provided to foster a deeper understanding of grade-level mathematics.

In this packet, you will find links to videos, links to online practice, and pencil-and-paper practice problems. The Table of Contents shows day-by-day lessons from April 14th to June 19th. We encourage you to take every advantage of the material in this packet.

Daily lesson guidance can be found in the table of contents below. Each day has been designed to provide you access to materials from Khan Academy and the academic packet. Each lesson has this structure:

Watch: Khan Academy (if internet access is available)	Practice: Khan Academy (if internet access is available)	Pencil & Paper Practice: Academic Packet
Watch and take notes on the lesson video on Khan Academy	Complete the practice exercises on Khan Academy	Complete the pencil and paper practice.

If one-on-one, live support is required, please feel free to call the **Homework Hotline** at **1-833-466-3978**. Please check the Homework Hotline page for operating hours. We have DPSCD mathematics teachers standing by and are ready to assist.

We appreciate your continued dedication, support and partnership with Detroit Public Schools Community District and with your assistance we can press forward with our priority: Outstanding Achievement. Be safe. Be well!

Deputy Executive Director of K-12 Mathematics

Important Links and Information

Clever

Students access Clever by visiting www.clever.com/in/dpscd.

What are my username and password for Clever?
Students access Clever using their DPSCD login credentials. Usernames and passwords follow this structure:

Username: studentID@thedps.org
Ex. If Aretha Franklin is a DPSCD student with a student ID of 018765 her username would be 018765@thedps.org.

Password:
First letter of first name in upper case
First letter of last name in lower case
2-digit month of birth
2-digit year of birth
01 (male) or 02 (female)
For example: If Aretha Franklin's birthday is March 25, 1998, her password and password would be Af039802.

Accessing Khan Academy

To access Khan Academy, visit www.clever.com/in/dpscd. Once logged into Clever, select the Khan Academy button:

Khan Academy

Accessing Your CPM eBook

Students can access their CPM eBook in two ways:

Option 1: Access the eBook through Clever

1. Visit www.clever.com/in/dpscd. Login using your DPSCD credentials above.
2. Click on the CPM icon:

Option 2: Visit http://open-ebooks.cpm.org/

1. Visit the website listed above.
2. Click "I agree"
3. Select the CPM Geometry eBook:

CC Geometry

Desmos Online Graphing Calculator

Access to a free online graphing and scientific calculator can be found at https://www.desmos.com/calculator.

Table of Contents

In the following table, you will find the table of contents and schedule for the week of April 13, 2020 through the week of June 15, 2020.

Week of **4/20-4/24**	Day 1	5.1.1-5.1.3 More Trigonometry	Solving for Angles and Inverse Trigonometric Functions (Article - Not a Video)	Solving for Angles Practice	Problems 25-28
	Day 2	5.1.1-5.1.3 More Trigonometry	Using Trig to Solve Word Problems	Trig Word Problems Practice	Problems 29-34
	Day 3	5.2.1-5.2.2 Special Right Triangles	Special Right Triangles Part 1	Special Right Triangles Practice	Problems 1-3
	Day 4	5.2.1-5.2.2 Special Right Triangles	Special Right Triangles Part 2	Special Right Triangles Practice	Problems 4-6
	Day 5	5.2.1-5.2.2 Special Right Triangles	30 60 90 Triangles	Special Right Triangles Practice	Problems 7-10

Week of 4/27-05/01	Day 1	5.3.1-5.3.3 Non-Right Triangles	Law of Sines	Law of Sines	Problems 1-6
	Day 2	5.3.1-5.3.3 Non-Right Triangles	Law of Sines	Law of Sines	Problems 7-12
	Day 3	5.3.1-5.3.3 Non-Right Triangles	Law of Cosines	Law of Cosines	Problems 13-18
	Day 4	5.3.1-5.3.3 Non-Right Triangles	Law of Cosines	Law of Cosines	Problems 19-24
	Day 5	5.3.1-5.3.3 Non-Right Triangles	Law of Sines and Cosines	Law of Sines and Cosines	Problems 25-30

Week of 05/04-05/08	Day 1	5.3.1-5.3.3 Non-Right Triangles	Law of Sines and Cosines	Law of Sines and Cosines	Problems 31-36
	Day 2	5.3.4-5.3.5 Triangle Ambiguity	Triangle inequality Theorem	Triangle Inequality Theorem	Problems 1-3
	Day 3	5.3.4-5.3.5 Triangle Ambiguity	Triangle inequality Theorem	Triangle Inequality Theorem	Problems 4-6
	Day 4	8.1.1-8.1.5 Polygons	Interior Angle Sum Theorem	Interior Angle Sum	Problems 1-6
	Day 5	8.1.1-8.1.5 Polygons	Interior Angle Sum Theorem	Angles of a Polygon	Problems 7-12

Week of 05/11-05/15	Day 1	8.1.1-8.1.5 Polygons	Exterior Angle Sum Theorem	Interior and Exterior Angles of a Polygon	Problems 13-18
	Day 2	8.1.1-8.1.5 Polygons	Exterior Angle Sum Theorem	Interior and Exterior Angles of a Polygon	Problems 19-24
	Day 3	8.1.1-8.1.5 Polygons	Area of a Polygon	Area of a Polygon	Problems 25-30
	Day 4	8.2.1-8.2.2 Area Ratios of Similar Figures	Area: similarity and congruence	Area of a Polygon	Problems 1-3

	Day 5	8.2.1-8.2.2 Area Ratios of Similar Figures	Area: similarity and congruence	Area of a Polygon	Problems 4-6
Week of 05/18-05/22	Day 1	8.3.1-8.3.3 Circumference and Area of Circles	Radius, diameter, circumference & π	Radius and diameter	Problems 1-7
	Day 2	8.3.1-8.3.3 Circumference and Area of Circles	Radius, diameter, circumference & π	Area of a circle	Problems 8-11
	Day 3	8.3.1-8.3.3 Circumference and Area of Circles	Radius, diameter, circumference & π	Area of a circle	Problems 12-15

	Day 4	8.3.1-8.3.3 Circumference and Area of Circles	Radius, diameter, circumference & π	Area of parts of a circle	Problems 16-22
	Day 5	8.3.1-8.3.3 Circumference and Area of Circles	Radius, diameter, circumference & π	Area of parts of a circle	Problems 23-28
Week of 05/25-05/29	Day 1	Holiday	N/A	N/A	N/A
	Day 2	9.1.1-9.1.5 Solids and Their Measurements	Measuring volume and surface area	Measuring volume and surface area Practice	Problems 1-6
	Day 3	9.1.1-9.1.5 Solids and Their Measurements	Measuring volume and surface area	Measuring volume and surface area Practice	Problems 7-11

	Day 4	9.1.1-9.1.5 Solids and Their Measurements	Measuring volume and surface area	Measuring volume and surface area Practice	Problems 12-13
	Day 5	9.1.1-9.1.5 Solids and Their Measurements	Measuring volume and surface area	Measuring volume and surface area Practice	Problems 14-20
Week of 06/01-06/05	Day 1	9.1.1-9.1.5 Solids and Their Measurements	Measuring volume and surface area	Measuring volume and surface area Practice	Problems 21-26
	Day 2	9.1.1-9.1.5 Solids and Their Measurements	Measuring volume and surface area	Measuring volume and surface area Practice	Problems 27-32

	Day 3	10.2.1-10.2.3 Conditional Probability and Two-Way Tables	Calculating conditional probability	Calculating Conditional Probability Practice	Problems 1-3
	Day 4	10.2.1-10.2.3 Conditional Probability and Two-Way Tables	Conditional probability and independence	Calculating Conditional Probability Practice	Problems 4-6
	Day 5	10.2.1-10.2.3 Conditional Probability and Two-Way Tables	Conditional probability and independence	Calculating Conditional Probability Practice	Problems 7-10

Week of 06/08-06/12	Day 1	11.1.1 – 11.1.5: Surface Area and Volume of Spheres, Cones, and Pyramids	Volume and Surface Area	Volume and Surface Area Practice	Problems 1-7
	Day 2	11.1.1 – 11.1.5: Surface Area and Volume of Spheres, Cones, and Pyramids	Volume and Surface Area	Volume and Surface Area Practice	Problems 8-18
	Day 3	11.1.1 – 11.1.5: Surface Area and Volume of Spheres, Cones, and Pyramids	Volume and Surface Area	Volume and Surface Area Practice	Problems 19-27
	Day 4	11.1.1 – 11.1.5: Surface Area and Volume of Spheres, Cones, and Pyramids	Volume and Surface Area	Volume and Surface Area Practice	Problems 28-33

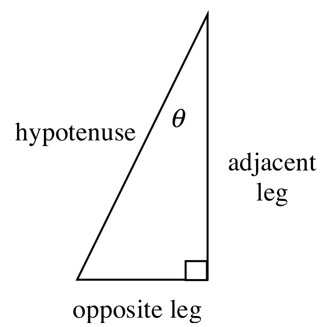
	Day 5	11.1.1 – 11.1.5: Surface Area and Volume of Spheres, Cones, and Pyramids	Volume and Surface Area	Volume and Surface Area Practice	Problems 34-39
Week of 06/15-06/19	Day 1	11.1.1 – 11.1.5: Surface Area and Volume of Spheres, Cones, and Pyramids	Volume and Surface Area	Volume and Surface Area Practice	Problems 40-51
	Day 2	11.1.1 – 11.1.5: Surface Area and Volume of Spheres, Cones, and Pyramids	Volume and Surface Area	Volume and Surface Area Practice	Problems 52-63
	Day 3	10.3.1-10.3.5 Principles of Counting	Counting, permutations, and combinations	Counting, permutations, and combinations Practice	Problems 1-12

	Day 4	10.3.1-10.3.5 Principles of Counting	Counting, permutations, and combinations	Counting, permutations, and combinations Practice	Problems 13-20
	Day 5	10.3.1-10.3.5 Principles of Counting	Counting, permutations, and combinations	Counting, permutations, and combinations Practice	Problems 21-25

We next introduce two more trigonometric ratios: sine and cosine. Both of them are used with acute angles of right triangles, just as the tangent ratio is. Using the diagram below:

$\sin\theta = \frac{\text{opposite leg}}{\text{hypotenuse}}$ $\quad\cos\theta = \frac{\text{adjacent leg}}{\text{hypotenuse}}$

and from Chapter 4: $\tan\theta = \frac{\text{opposite leg}}{\text{adjacent leg}}$



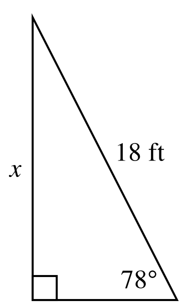
Note: If you decide to use the other acute angle in the triangle, then the names of the legs switch places. The opposite leg is always across the triangle from the acute angle you are using.

See the Math Notes boxes in Lessons 5.1.2 and 5.1.4.

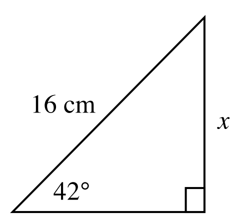
Example 1

Use the sine ratio to find the length of the unknown side in each triangle below.

a.



b.



The sine of the angle is the ratio $\frac{\text{opposite leg}}{\text{hypotenuse}}$. For part (a) we will use the 78° as θ. From the 78° angle, we find which side of the triangle is the opposite leg and which side is the hypotenuse. The hypotenuse is always the longest side, and it is always opposite the right angle. In this case, it is 18. From the 78° angle, the opposite leg is the side labeled x. Now we can write the equation at right and solve it.

$$\sin 78° = \tfrac{x}{18} \left(\frac{\text{opposite}}{\text{hypotenuse}}\right)$$
$$18 \sin 78° = x$$
$$x \approx 17.61 \text{ ft}$$

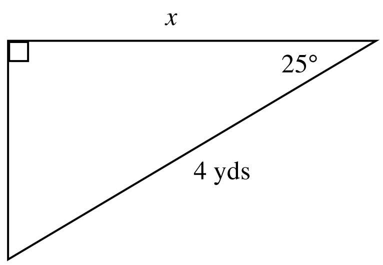
In part (b), from the 42° angle, the opposite leg is x and the hypotenuse is 16. We can write and solve the equation at right. Note: In most cases, it is most efficient to wait until the equation has been solved for x, then use your calculator to combine the values, as shown in these examples.

$$\sin 42° = \tfrac{x}{16}$$
$$16(\sin 42°) = x$$
$$x \approx 10.71 \text{ cm}$$

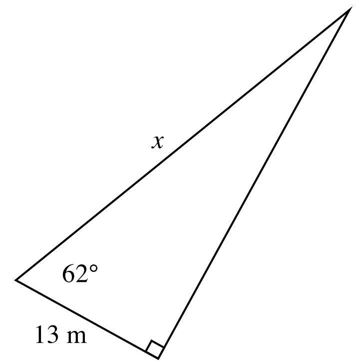
Example 2

Use the cosine ratio to find the length of the unknown side in each triangle below.

a.



b.



Just as before, we set up an equation using the cosine ratio, $\frac{\text{adjacent leg}}{\text{hypotenuse}}$. Remember that you can always rotate the page, or trace and rotate the triangle, if the figure's orientation is causing confusion. The key to solving these problems is recognizing which side is adjacent, which is opposite, and which is the hypotenuse. (See the box above Example 1 for this information.) For part (a), the angle is 25°, so we can write and solve the equation at right.

$$\cos 25° = \frac{x}{4} \quad \left(\frac{\text{adjacent}}{\text{hypotenuse}}\right)$$
$$4(\cos 25°) = x$$
$$x \approx 3.63 \text{ yds}$$

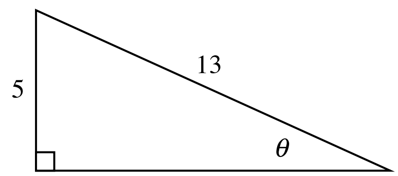
In part (b), from the 62° angle, the adjacent leg is 13 and the hypotenuse is x. This time, our variable will be in the denominator. As we saw in earlier chapters, this will add one more step to the solution.

$$\cos 62° = \frac{13}{x}$$
$$x \cos 62° = 13$$
$$x = \frac{13}{\cos 62°} \approx 27.69 \text{ m}$$

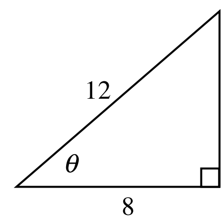
Example 3

In each triangle below, use the inverse trigonometry buttons on your calculator to find the measure of the angle θ to the nearest hundredth.

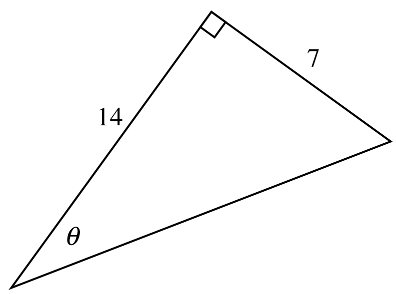
a.



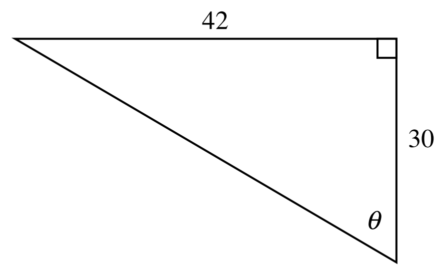
b.



c.



d.



For each of these problems you must decide whether you will be using sine, cosine, or tangent to find the value of θ. In part (a), if we are standing at the angle θ, then 5 is the length of the opposite leg and 13 is the length of the hypotenuse. This tells us to use the sine ratio. For the best accuracy, enter the ratio, not its decimal approximation.

$$\sin\theta = \frac{5}{13}$$
$$\sin\theta \approx 0.385$$

To find the value of θ, find the button on the calculator that says $\sin^{-1}$. (Note: Calculator sequences shown are for most graphing calculators. Some calculators use a different order of keystrokes.) This is the "inverse sine" key, and when a ratio is entered, this button tells you the measure of the angle that has that sine ratio. Here we find $\sin^{-1}\frac{5}{13} \approx 22.62°$ by entering "2nd," "sin," (5 ÷ 13), "enter." Be sure to use parentheses as shown.

In part (b), 8 is the length of the adjacent leg and 12 is the length of the hypotenuse. This combination of sides fits the cosine ratio. We use the $\cos^{-1}$ button to find the measure of θ by entering the following sequence on the calculator: "2nd," "cos," (8 ÷ 12), "enter."

$$\cos\theta = \frac{8}{12}$$
$$\cos\theta \approx 0.667$$
$$\theta = \cos^{-1}\frac{8}{12}$$
$$\theta \approx 48.19°$$

In part (c), from θ, 7 is the length of the opposite leg and 14 is the length of the adjacent leg. These two sides fit the tangent ratio. As before, you need to find the $\tan^{-1}$ button on the calculator.

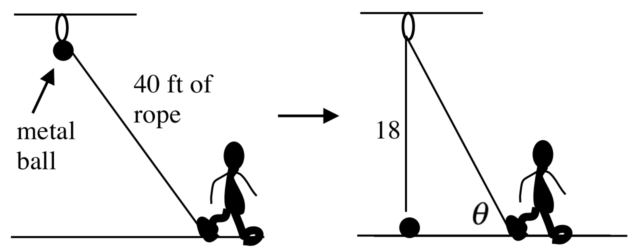
$$\tan\theta = \frac{7}{14} = 0.5$$
$$\tan\theta = 0.5$$
$$\theta = \tan^{-1} 0.5 \approx 26.57°$$

If we are standing at the angle θ in part (d), 42 is the length of the opposite leg while 30 is the length of the adjacent leg. We will use the tangent ratio to find the value of θ.

$$\tan\theta = \frac{42}{30} = 1.4$$
$$\tan\theta = 1.4$$
$$\theta = \tan^{-1} 1.4 \approx 54.46°$$

Example 4

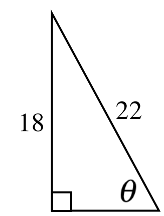
Kennedy is standing on the end of a rope that is 40 feet long and threaded through a pulley. The rope is holding a large metal ball 18 feet above the floor. Kennedy slowly slides her feet closer to the pulley to lower the ball. When the ball hits the floor, what angle (θ) does the rope make with the floor where it is under her foot?



As always, we must draw a picture of this situation to determine what we must do. We start with a picture of the beginning situation, before Kennedy has started lowering the ball. The second picture shows the situation once the ball has reached the floor. We want to find the angle θ. You should see a right triangle emerging, made of the rope and the floor. The 40-foot rope makes up two sides of the triangle: 18 feet is the length of the leg opposite θ, and the rest of the rope, 22 feet of it, is the hypotenuse. With this information, draw one more picture. This one will show the simple triangle that represents this situation.

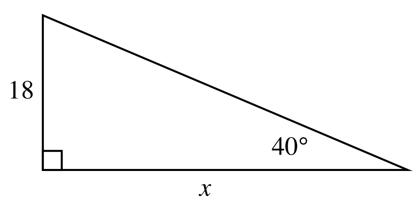
From θ, we have the lengths of the opposite leg and the hypotenuse. This tells us to use the sine ratio.

$$\sin\theta = \tfrac{18}{22}$$
$$\theta = \sin^{-1}\tfrac{18}{22}$$
$$\theta \approx 54.9°$$



Problems

Using the tangent, sine, and cosine buttons on your calculator, calculate the value of x to the nearest hundredth.

1.


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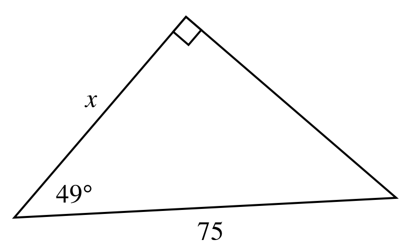

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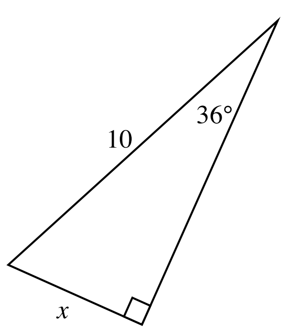

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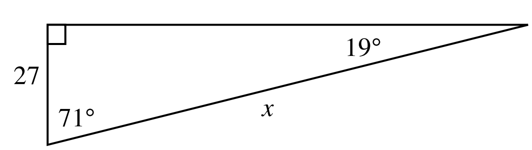

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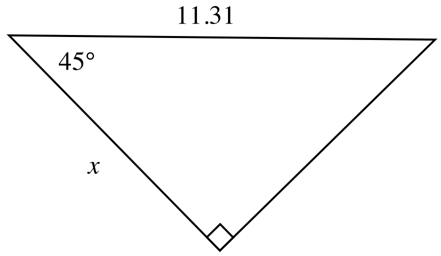

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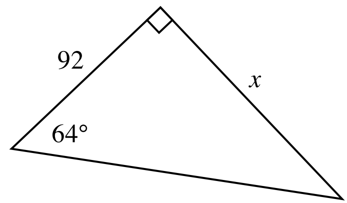
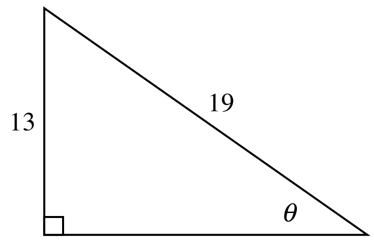


Using the $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ buttons on your calculator, calculate the value of θ to the nearest hundredth.

7.



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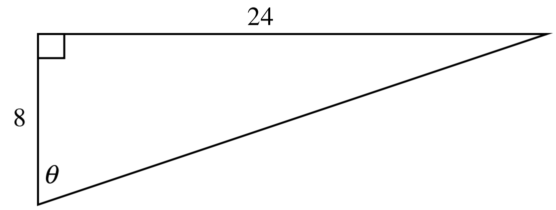

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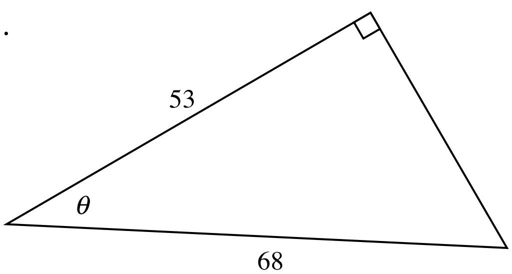

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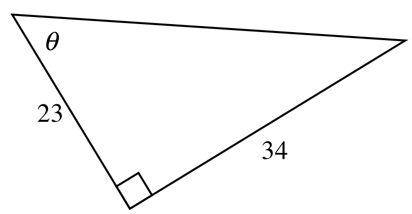

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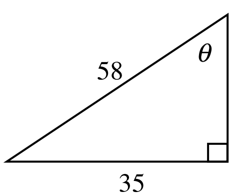

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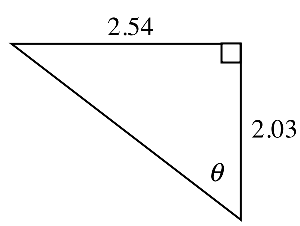
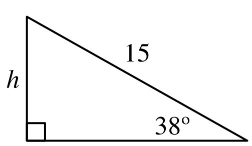


Use trigonometric ratios to solve for the variable in each figure below.

13.



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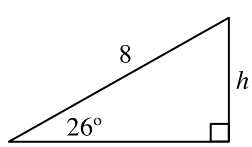

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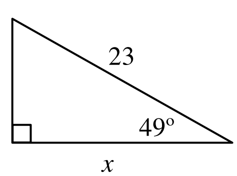

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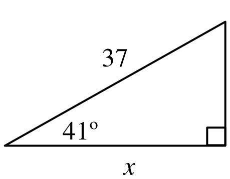

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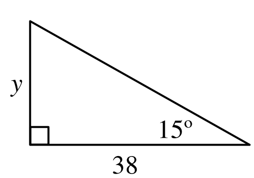

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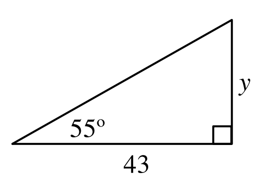
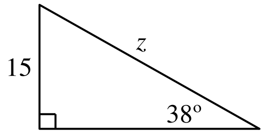
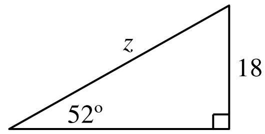
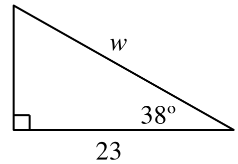
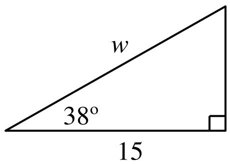
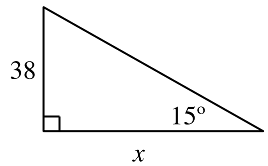


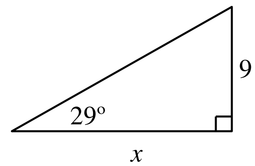
19. 

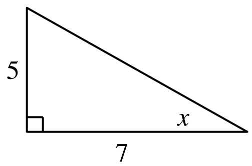
20. 

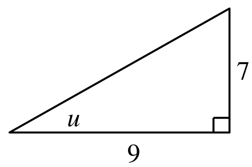
21. 

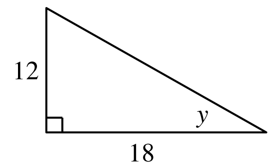
22. 

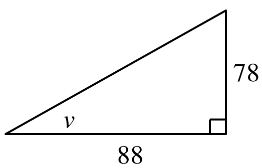
23. 

24. 

25. 

26. 

27. 

28. 

Draw a diagram and use trigonometric ratios to solve each of the following problems.

29. Juanito is flying a kite at the park and realizes that all 500 feet of string are out. Margie measures the angle of the string with the ground with her clinometer and finds it to be 42.5º. How high is Juanito's kite above the ground?

30. Nell's kite has a 350 foot string. When it is completely out, Ian measures the angle to be 47.5º. How far would Ian need to walk to be directly under the kite?

31. Mayfield High School's flagpole is 15 feet high. Using a clinometer, Tamara measured an angle of 11.3º to the top of the pole. Tamara is 62 inches tall. How far from the flagpole is Tamara standing?

32. Tamara took another sighting of the top of the flagpole from a different position. This time the angle is 58.4º. If everything else is the same, how far from the flagpole is Tamara standing?

33. Standing 140 feet from the base of a building, Alejandro uses his clinometer to site the top of the building. The reading on his clinometer is 42°. If his eyes are 6 feet above the ground, how tall is the building?

34. An 18 foot ladder rests against a wall. The base of the ladder is 8 feet from the wall. What angle does the ladder make with the ground?

Answers

1. $\tan 40° = \frac{18}{x}$, $x \approx 21.45$

2. $\cos 49° = \frac{x}{75}$, $x \approx 49.20$

3. $\sin 36° = \frac{x}{10}$, $x \approx 5.88$

4. $\sin 19° = \frac{27}{x}$ or $\cos 71° = \frac{27}{x}$, $x \approx 82.93$

5. $\cos 45° = \frac{x}{11.31}$, $x \approx 8.00$

6. $\tan 64° = \frac{x}{92}$, $x \approx 188.63$

7. $\sin\theta = \frac{13}{19}$, $\theta \approx 43.17^{\circ}$

8. $\tan\theta = \frac{24}{8}$, $\theta \approx 71.57^{\circ}$

9. $\cos\theta = \frac{53}{68}$, $\theta \approx 38.79^{\circ}$

10. $\tan\theta = \frac{34}{23}$, $\theta \approx 55.92^{\circ}$

11. $\sin\theta = \frac{35}{58}$, $\theta \approx 37.12^{\circ}$

12. $\tan\theta = \frac{2.54}{2.03}$, $\theta \approx 51.37^{\circ}$

13. $h = 15 \sin 38^{\circ} \approx 9.24$

14. $h = 8 \sin 26^{\circ} \approx 3.51$

15. $x = 23 \cos 49^{\circ} \approx 15.09$

16. $x = 37 \cos 41^{\circ} \approx 27.92$

17. $y = 38 \tan 15^{\circ} \approx 10.18$

18. $y = 43 \tan 55^{\circ} \approx 61.41$

19. $z = \frac{15}{\sin 38^{\circ}} \approx 24.364$

20. $z = \frac{18}{\sin 52^{\circ}} \approx 22.84$

21. $w = \frac{23}{\cos 38^{\circ}} \approx 29.19$

22. $w = \frac{15}{\cos 38^{\circ}} \approx 19.04$

23. $x = \frac{38}{\tan 15^{\circ}} \approx 141.82$

24. $x = \frac{91}{\tan 29^{\circ}} \approx 164.17$

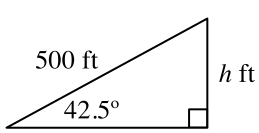
25. $x = \tan^{-1} \frac{5}{7} \approx 35.54^{\circ}$

26. $u = \tan^{-1} \frac{7}{9} \approx 37.88^{\circ}$

27. $y = \tan^{-1} \frac{12}{18} \approx 33.69^{\circ}$

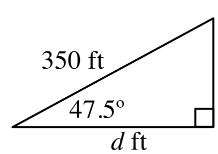
28. $y = \tan^{-1} \frac{78}{88} \approx 41.55^{\circ}$

29.



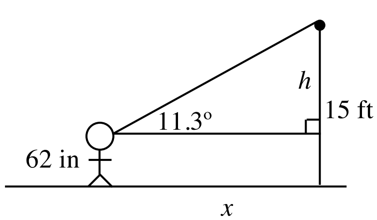
$\sin 42.5^{\circ} = \frac{h}{500}$
$h = 500 \sin 42.5^{\circ} \approx 337.8$ ft

30.



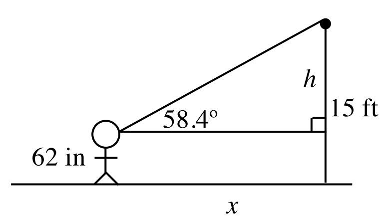
$\cos 47.5^{\circ} = \frac{d}{350}$
$d = 350 \cos 47.5^{\circ} \approx 236.5$ ft

31.



15 feet = 180 inches,
$180'' - 62'' = 118'' = h$
$x \approx 590.5$ inches or 49.2 ft.

32.



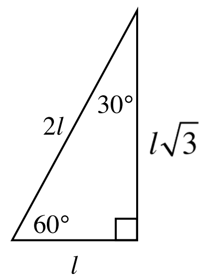
$h = 118''$, $\tan 58.4^{\circ} = \frac{118''}{x}$,
$x \tan 58.4 = 118''$, $x = \frac{118''}{\tan 58.4^{\circ}}$
$x \approx 72.6$ inches or 6.1 ft.

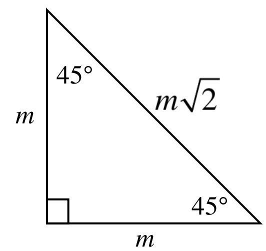
33. $\tan 42° = \frac{h}{140}$, $h + 6 \approx 132$ feet

34. $\cos\theta = \frac{8}{18}$, $\theta \approx 63.61^{\circ}$

SPECIAL RIGHT TRIANGLES 5.2.1 – 5.2.2

There are two special right triangles that occur often in mathematics: the 30°-60°-90° triangle and the 45°-45°-90° triangle. By AA ~, all 30°-60°-90° triangles are similar to each other, and all 45°-45°-90° triangles are similar to each other. Consequently, for each type of triangle, the sides are proportional. The sides of these triangles follow these patterns.





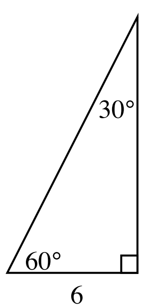
Another short cut in recognizing side lengths of right triangles are Pythagorean Triples. The lengths 3, 4, and 5 are sides of a right triangle (Note: You can verify this with the Pythagorean Theorem) and the sides of all triangles similar to the 3-4-5 triangle will have sides that form Pythagorean Triples (6-8-10, 9-12-15, etc). Another common Pythagorean Triple is 5-12-13.

See the Math Notes boxes in Lessons 5.2.1 and 5.3.1.

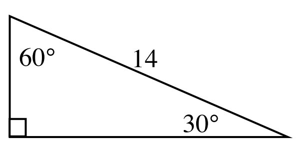
Example 1

The triangles below are either a 30°-60°-90° triangle or a 45°-45°-90° triangle. Decide which it is and find the lengths of the other two sides based on the pattern for that type of triangle.

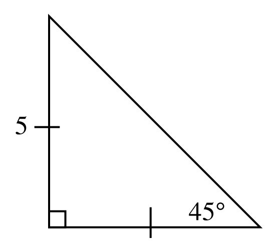
a.



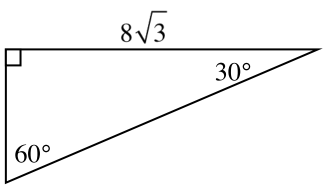
b.



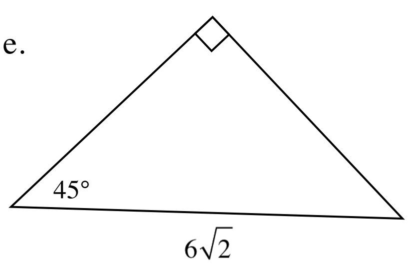
c.



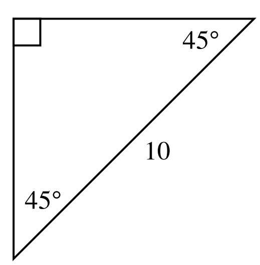
d.



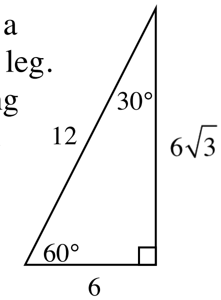
e.



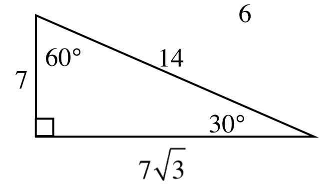
f.



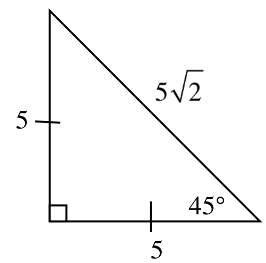
In part (a), this is a 30°-60°-90° triangle, so its sides will fit the pattern for such a triangle. The pattern tells us that the hypotenuse is twice the length of the short leg. Since the short leg has a length of 6, the hypotenuse has a length of 12. The long leg is the length of the short leg times $\sqrt{3}$, so the long leg has a length of $6\sqrt{3}$.



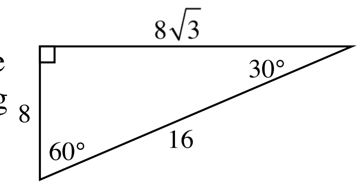
In part (b), we have a 30°-60°-90° triangle again, but this time we know the length of the hypotenuse. Following the pattern, this means the length of the short leg is half the hypotenuse: 7. As before, we multiply the length of the short leg by $\sqrt{3}$ to get the length of the long leg: $7\sqrt{3}$.



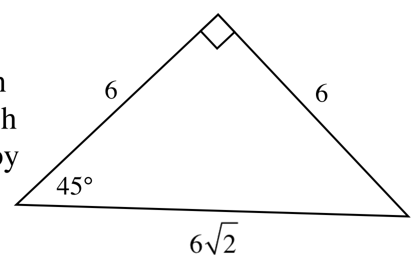
The triangle in part (c) is a 45°-45°-90° triangle. The missing angle is also 45°; you can verify this by remembering the sum of the angles of a triangle is 180°. The legs of a 45°-45°-90° triangle are equal in length (it is isosceles) so the length of the missing leg is also 5. To find the length of the hypotenuse, we multiply the leg's length by $\sqrt{2}$. Therefore the hypotenuse has length $5\sqrt{2}$.



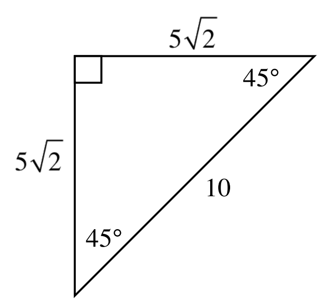
We have another 30°-60°-90° triangle in part (d). This time we are given the length of the long leg. To find the short leg, we *divide* the length of the long leg by $\sqrt{3}$. Therefore, the length of the short leg is 8. To find the length of the hypotenuse, we double the length of the short leg, so the hypotenuse is 16.



The triangle in part (e) is a 45°-45°-90° triangle, and we are given the length of the hypotenuse. To find the length of the legs (which are equal in length), we will divide the length of the hypotenuse by $\sqrt{2}$. Therefore, each leg has length 6.



If you understand what was done in each of the previous parts, part (f) is no different from the rest. This is a 45º-45º-90º triangle, and we are given the length of the hypotenuse. However, we are used to seeing the hypotenuse of a 45º-45º-90º triangle with a $\sqrt{2}$ attached to it. In the last part when we were given the length of the hypotenuse, we divided by $\sqrt{2}$ to find the length of the legs, and this time we do the same thing.



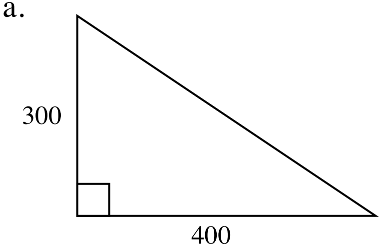
$$\frac{10}{\sqrt{2}} = \frac{10}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$$

Note: Multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$ is called rationalizing the denominator. It is a technique to remove the radical from the denominator.

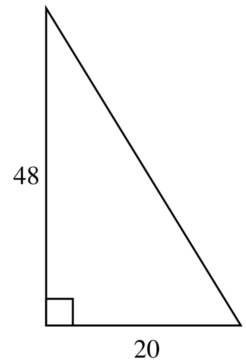
Example 2

Use what you know about Pythagorean Triples and similar triangles to fill in the missing lengths of sides below.

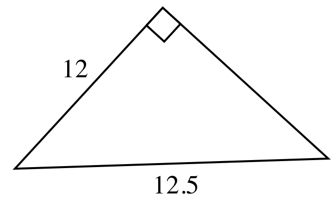
a.



b.



c.

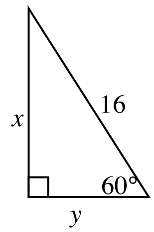


There are a few common Pythagorean Triples that students should recognize; 3–4–5, 5–12–13, 8–15–17, and 7–24–25 are the most common. If you forget about a particular triple or do not recognize one, you can always find the unknown side by using the Pythagorean Theorem if two of the sides are given. In part (a), this is a multiple of a 3–4–5 triangle. Therefore the length of the hypotenuse is 500. In part (b), we might notice that each leg has a length that is a multiply of four. Knowing this, we can rewrite them as 48 = 4(12), and 20 = (4)(5). This is a multiple of a 5–12–13 triangle, the multiplier being 4. Therefore, the length of the hypotenuse is 4(13) = 42. In part (c), do not let the decimal bother you. In fact, since we are working with Pythagorean Triples and their multiples, double both sides to create a similar triangle. This eliminates the decimal. That makes the leg 24 and the hypotenuse 25. Now we recognize the triple as 7–24–25. Since the multiple is 0.5, the length of the other leg is 3.5.

Problems

Identify the special triangle relationships. Then solve for x, y, or both.

1.



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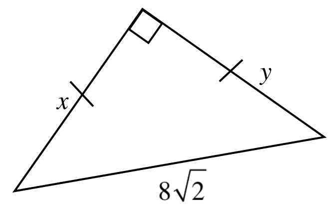
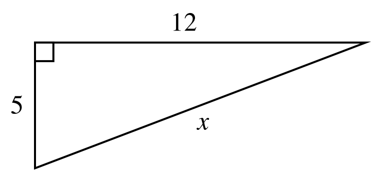


3.



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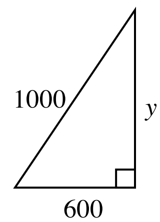

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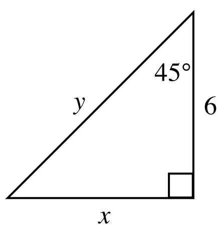

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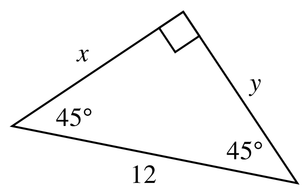

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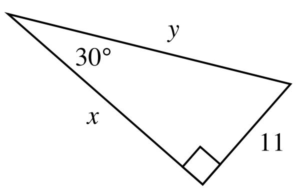

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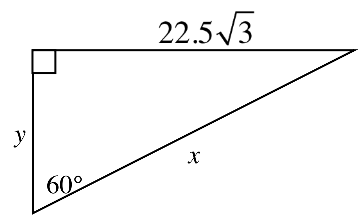
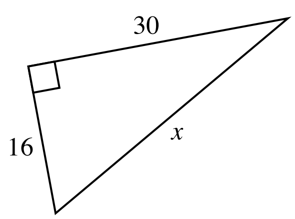
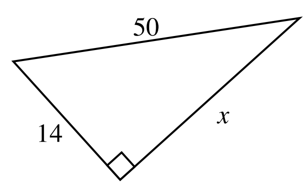


9.



10.



Answers

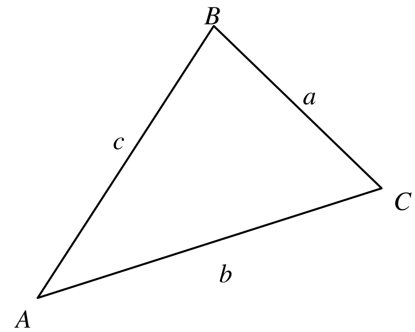
1. $x = 8\sqrt{3}$, $y = 8$
2. $x = y = 8$
3. $x = 13$
4. $y = 800$
5. $x = 6,\ y = 6\sqrt{2}$
6. $x = y = \frac{12}{\sqrt{2}} = 6\sqrt{2}$
7. $x = 11\sqrt{3}$, $y = 22$
8. $x = 45$, $y = 22.5$
9. $x = 34$
10. $x = 48$

NON-RIGHT TRIANGLES 5.3.1 – 5.3.3

Students have several tools for finding parts of right triangles, including the Pythagorean Theorem, the tangent ratio, the sine ratio, and the cosine ratio. These relationships only work, however, with *right* triangles. What if the triangle is not a right triangle? Can we still calculate lengths and angles with trigonometry from certain pieces of information? Yes, by using two laws, the Law of Sines and the Law of Cosines that state:

Law of Sines

$$\frac{\sin(m\angle A)}{a} = \frac{\sin(m\angle B)}{b}$$
$$\frac{\sin(m\angle B)}{b} = \frac{\sin(m\angle C)}{c}$$
$$\frac{\sin(m\angle A)}{a} = \frac{\sin(m\angle C)}{c}$$



Law of Cosines

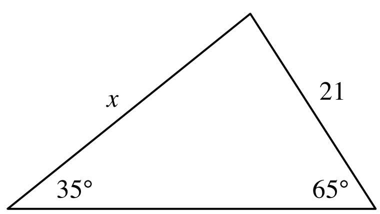
$$c^2 = a^2 + b^2 - 2ab\cos C \qquad b^2 = a^2 + c^2 - 2ac\cos B \qquad a^2 = b^2 + c^2 - 2bc\cos A$$

See the Math Notes boxes in Lessons 5.3.2 and 5.3.3.

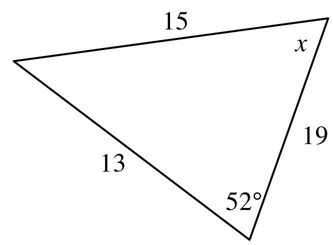
Example 1

Using the Law of Sines, calculate the value of x.

a.



b.



We will set up ratios that are equal according to the Law of Sines. The ratio compares the sine of the measure of an angle to the length of the side opposite that angle. In part (a), 21 is the length of the side opposite the 35° angle, while x is the length of the side opposite the 65° angle. The proportion is shown at right. To solve the proportion, we cross multiply, and solve for x. We can use the Law of Sines to find the measure of an angle as well. In part (b), we again write a proportion using the Law of Sines.

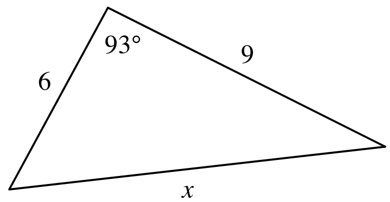
$$\frac{\sin 35°}{21} = \frac{\sin 65°}{x}$$
$$x\sin 35° = 21\sin 65°$$
$$x = \frac{21\sin 65°}{\sin 35°}$$
$$x \approx 33.18$$

$$\frac{\sin x}{13} = \frac{\sin 52°}{15}$$
$$15\sin x = 13\sin 52°$$
$$\sin x = \frac{13\sin 52°}{15}$$
$$\sin^{-1} x = 13\sin 52° \div 15$$
$$x \approx 43.07°$$

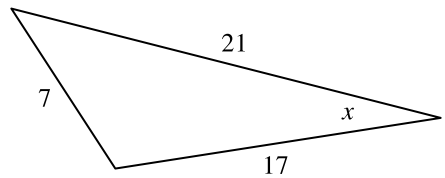
Example 2

Use the Law of Cosines to solve for x in the triangles below.

a.



b.



The Law of Cosines does not use ratios, as the Law of Sines does. Rather, it uses a formula somewhat similar to the Pythagorean Theorem. For part (a) the formula gives us the equation and solution shown below.

$$\begin{aligned} x^2 &= 6^2 + 9^2 - 2(6)(9)\cos 93° \\ x^2 &\approx 36 + 81 - 108(-0.052) \\ x^2 &\approx 117 + 5.612 \\ x^2 &\approx 122.612 \\ x &\approx 11.07 \end{aligned}$$

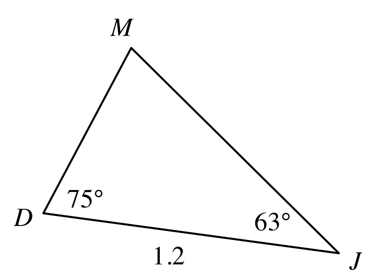
Just as with the Law of Sines, we can use the Law of Cosines to find the measures of angles as well as side lengths. In part (b) we will use the Law of Cosines to find the measure of angle x. From the law we can write the equation and solution shown below.

$$\begin{aligned} 7^2 &= 17^2 + 21^2 - 2(17)(21)\cos x \\ 49 &= 289 + 441 - 714\cos x \\ 49 &= 730 - 714\cos x \\ -681 &= -714\cos x \\ \tfrac{-681}{-714} &= \cos x \\ x &\approx 17.49° \text{ (using } \cos^{-1} x) \end{aligned}$$

Example 3

Marisa's, June's, and Daniel's houses form a triangle. The distance between June's and Daniel's houses is 1.2 km. Standing at June's house, the angle formed by looking out to Daniel's house and then to Marisa's house is 63°. Standing at Daniel's house, the angle formed by looking out to June's house and then to Marisa's house is 75°. What is the distance between all of the houses?

The trigonometry ratios and laws are very powerful tools in real world situations. As with any application, the first step is to draw a picture of the situation. We know the three homes form a triangle, so we start with that. We already know one distance: the distance from June's house to Daniel's house. We write 1.2 as the length of the side from D to J. We also know that $m\angle J = 63°$ and $m\angle D = 75°$, and can figure out that $m\angle M = 42°$. We are trying to find the lengths of $\overline{DM}$ and $\overline{MJ}$. To do this, we will use the Law of Sines.



MJ:

$$\begin{aligned} \tfrac{\sin 75°}{MJ} &= \tfrac{\sin 42°}{1.2} \\ 1.2\sin 75° &= (MJ)\sin 42° \\ \tfrac{1.2\sin 75°}{\sin 42°} &= MJ \\ MJ &\approx 1.73 \text{ km} \end{aligned}$$

DM:

$$\begin{aligned} \tfrac{\sin 63°}{DM} &= \tfrac{\sin 42°}{1.2} \\ 1.2\sin 63° &= (DM)\sin 42° \\ \tfrac{1.2\sin 63°}{\sin 42°} &= DM \\ DM &\approx 1.60 \text{ km} \end{aligned}$$

Therefore the distances between the homes are: From Marisa's to Daniel's: 1.6 km, from Marisa's to June's: 1.73 km, and from Daniel's to June's: 1.2 km.

Problems

Use the tools you have for triangles to solve for x, y, or θ. Round all answers to the nearest hundredth.

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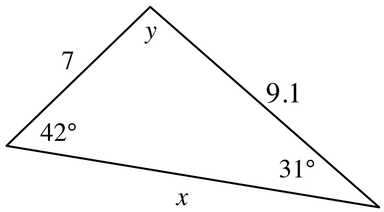

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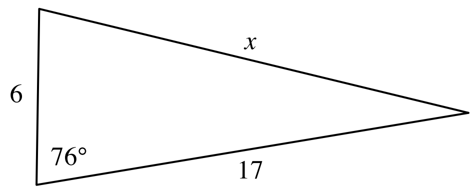

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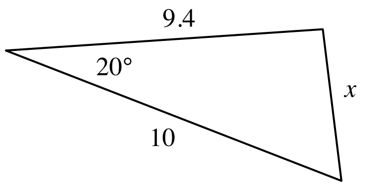

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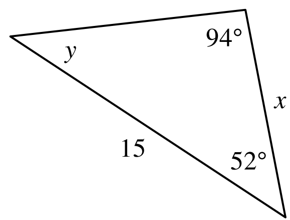

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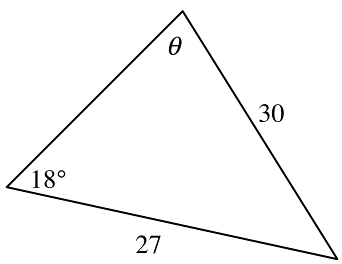

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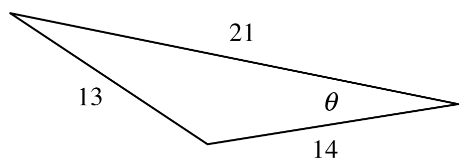

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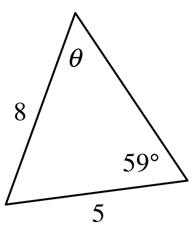

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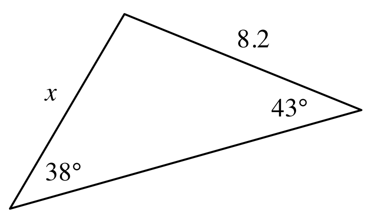

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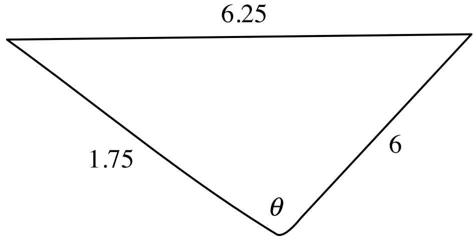

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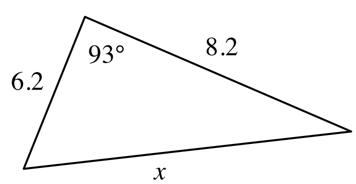
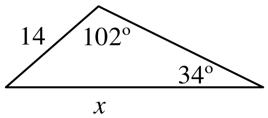
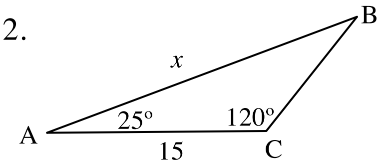
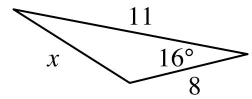
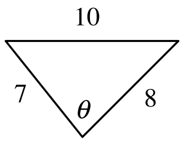


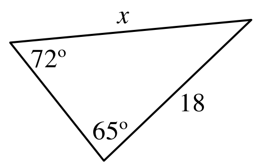
Use the Law of Sines or the Law of Cosines to find the required part of the triangle.

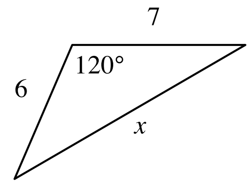
11. 

12. 

13. 

14. 

15. 

16. 

Draw and label a triangle similar to the one in the examples. Use the given information to find the required part(s).

17. $m\angle\text{A} = 40^\circ$, $m\angle\text{B} = 88^\circ$, $a = 15$.
Find b.

18. $m\angle\text{B} = 75^\circ$, $a = 13$, $c = 14$.
Find b.

19. $m\angle\text{B} = 50^\circ$, $m\angle\text{C} = 60^\circ$, $b = 9$.
Find a.

20. $m\angle\text{A} = 62^\circ$, $m\angle\text{C} = 28^\circ$, $c = 24$.
Find a.

21. $m\angle\text{A} = 51^\circ$, $c = 8$, $b = 12$.
Find a.

22. $m\angle\text{B} = 34^\circ$, $a = 4$, $b = 3$.
Find c.

23. $a = 9$, $b = 12$, $c = 15$.
Find $m\angle\text{B}$.

24. $m\angle\text{B} = 96^\circ$, $m\angle\text{A} = 32^\circ$, $a = 6$.
Find c.

25. $m\angle\text{C} = 18^\circ$, $m\angle\text{B} = 54^\circ$, $b = 18$.
Find c.

26. $a = 15$, $b = 12$, $c = 14$.
Find $m\angle\text{C}$.

27. $m\angle\text{C} = 76^\circ$, $a = 39$. $\text{B} = 19$.
Find c.

28. $m\angle\text{A} = 30^\circ$, $m\angle\text{C} = 60^\circ$, $a = 8$.
Find b.

29. $a = 34$, $b = 38$, $c = 31$.
Find $m\angle\text{B}$.

30. $a = 8$, $b = 16$, $c = 7$.
Find $m\angle\text{C}$.

31. $m\angle\text{C} = 84^\circ$, $m\angle\text{B} = 23^\circ$, $c = 11$.
Find b.

32. $m\angle\text{A} = 36^\circ$, $m\angle\text{B} = 68^\circ$, $b = 8$.
Find a and c.

33. $m\angle\text{B} = 40^\circ$, $b = 4$, and $c = 6$.
Find a, $m\angle\text{A}$, and $m\angle\text{C}$.

34. $a = 2$, $b = 3$, $c = 4$.
Find $m\angle\text{A}$, $m\angle\text{B}$, and $m\angle\text{C}$.

35. Marco wants to cut a sheet of plywood to fit over the top of his triangular sandbox. One angle measures 38°, and it is between sides with lengths 14 feet and 18 feet. What is the length of the third side?

36. From the planet Xentar, Dweeble can see the stars Quazam and Plibit. The angle between these two sites is 22°. Dweeble knows that Quazam and Plibit are 93,000,000 miles apart. He also knows that when standing on Plibit, the angle made from Quazam to Xentar is 39°. How far is Xentar from Quazam?

Answers

1. $x \approx 13.00$, $y = 107^\circ$
2. $x \approx 16.60$
3. $x \approx 3.42$
4. $x \approx 8.41$, $y = 34^\circ$
5. $\theta \approx 16.15^\circ$
6. $\theta \approx 37.26^\circ$
7. $\theta \approx 32.39^\circ$
8. $x \approx 9.08$
9. $\theta = 90^\circ$
10. $x \approx 10.54$
11. 24.49
12. 22.6
13. 4.0
14. 83.3°
15. 17.15
16. 11.3
17. 23.32
18. 16.46
19. 11.04
20. 45.14
21. 9.34
22. 5.32 or 1.32
23. 53.13°
24. 8.92
25. 6.88
26. 61.28°
27. 39.03
28. 16
29. 71.38°
30. no triangle
31. 4.32
32. 5.07, 8.37
33. 5.66, 65.4°, 74.6°
34. 28.96°, 46.57°, 104.45°
35. ≈ 11.08 feet
36. ≈ 156,235,361 miles

Sometimes the information we know about sides and angle of a triangle is not enough to make one unique triangle. Sometimes a triangle may not even exist, as we saw when we studied the Triangle Inequality. When a triangle formed is not unique (that is, more than one triangle can be made with the given conditions) we call this **triangle ambiguity**. This happens when we are given two sides and an angle *not* between the two sides, known as SSA.

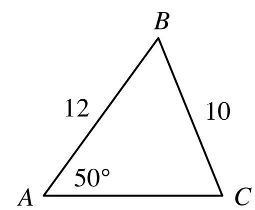
Example 1

In ΔABC, $m\angle A = 50°$, $AB = 12$, and $BC = 10$. Can you make a unique triangle? If so, find all the angle measures and side lengths for ΔABC. If not, show more than one triangle that meets these conditions.

As with many problems, we will first make a sketch of what the problem is describing.

Once we label the figure, we see that the information displays the SSA pattern mentioned above. It does seem as if this triangle can exist. First try to find the length of side AC. To do this we will use the Law of Cosines.

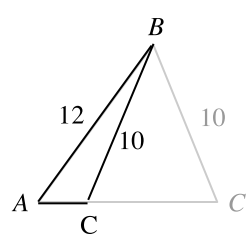
$$10^2 = 12^2 + x^2 - 2(12)(x)\cos 50°$$
$$100 = 144 + x^2 - 24x\cos 50°$$
$$100 \approx 144 + x^2 - 15.43x$$
$$x^2 - 15.43x + 44 \approx 0$$

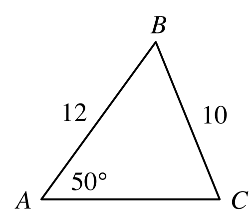


Now we have a type of equation we have not seen when solving this sort of problem. This is a quadratic equation. To solve it, we will use the Quadratic Formula (see the Math Notes box in Lesson 4.2.1). Recall that a quadratic equation may have two different solutions. We will use the formula and see what happens.

$$x^2 - 15.43x + 44 = 0$$
$$x = \frac{15.43 \pm \sqrt{15.43^2 - 4(1)(44)}}{2(1)}$$
$$\approx \frac{15.43 \pm \sqrt{238.08 - 176}}{2}$$
$$\approx \frac{15.43 \pm \sqrt{62.08}}{2}$$
$$\approx \frac{15.43 \pm 7.88}{2}$$
$$x \approx \frac{15.43 + 7.88}{2} \text{ or } x \approx \frac{15.43 - 7.88}{2}$$
$$x \approx 11.65 \text{ or } x \approx 3.78$$

Both of these answers are positive numbers, and could be lengths of sides of a triangle. So what happened? If we drew the triangle to scale, we would notice that although we drew the triangle with $\angle C$ acute, it does not have to be. Nothing in the information given says to draw the triangle this way. In fact, since there are no conditions on $m\angle B$ the side $\overline{BC}$ can swing as if it is on a hinge at $\angle B$. As you move $\overline{BC}$ along $\overline{AC}$, $\overline{BC}$ can intersect $\overline{AC}$ at two different places and still be 10 units long. In one arrangement, $\angle C$ is fairly small, while in the second arrangement, $\angle C$ is larger. (Note: The triangle formed with the two possible arrangements (the light grey triangle) is isosceles. From that you can conclude that the two possibilities for $\angle C$ are supplementary.)





Problems

Partial information is given about a triangle in each problem below. Solve for the remaining parts of the triangle, explain why a triangle does not exist, or explain why there is more than one possible triangle.

1. In ΔABC, $\angle A = 32^{\circ}$, $AB = 20$, and $BC = 12$.

2. In ΔXYZ, $\angle Z = 84^{\circ}$, $XZ = 6$, and $YZ = 9$.

3. In ΔABC, $m\angle A = m\angle B = 45^{\circ}$ and $AB = 7$.

4. In ΔPQR, $PQ = 15$, $\angle R = 28^{\circ}$, and $PR = 23$.

5. In ΔXYZ, $\angle X = 59^{\circ}$, $XY = 18$, and $YZ = 10$.

6. In ΔPQR, $\angle P = 54^{\circ}$, $\angle R = 36^{\circ}$, and $PQ = 6$.

Answers

1. Two triangles: AC ≈ 22.58, $m\angle B \approx 85.34^{\circ}$, $m\angle C \approx 62.66^{\circ}$ or $AC \approx 11.35$, $m\angle B \approx 30.04^{\circ}$, $m\angle C \approx 117.96^{\circ}$

2. One triangle: $XY \approx 10.28$, $m\angle X \approx 60.54^{\circ}$, $m\angle Y \approx 35.46^{\circ}$

3. One triangle: $m\angle C \approx 90^{\circ}$, $BC = AC = \frac{7\sqrt{2}}{2} \approx 4.95$

4. Two triangles: $QR \approx 30.725$, $m\angle Q \approx 46.04^{\circ}$, $m\angle P \approx 105.96^{\circ}$, or $QR \approx 9.895$, $m\angle Q \approx 133.96^{\circ}$, $m\angle P \approx 18.04^{\circ}$

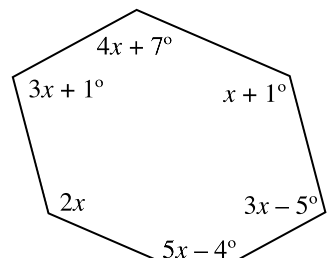
5. No triangle exists. Note: If you use Law of Cosines, you will have a negative number under the square root sign. This means there are no real number solutions.

6. One triangle: $m\angle Q = 90^{\circ}$, $QR \approx 8.26$, $PR \approx 10.21$

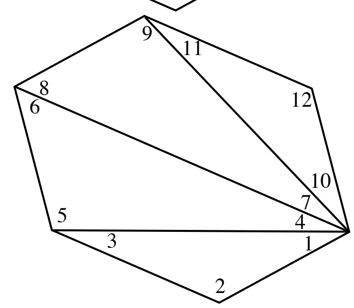
After studying triangles and quadrilaterals, students now extend their study to all polygons. A polygon is a closed, two-dimensional figure made of three or more non-intersecting straight line segments connected end-to-end. Using the fact that the sum of the measures of the angles in a triangle is 180°, students learn a method to determine the sum of the measures of the interior angles of any polygon. Next they explore the sum of the measures of the exterior angles of a polygon. Finally they use the information about the angles of polygons along with their Triangle Toolkits to find the areas of regular polygons.

See the Math Notes boxes in Lessons 8.1.1, 8.1.5, and 8.3.1.

Example 1



The figure at right is a hexagon. What is the sum of the measures of the interior angles of a hexagon? Explain how you know. Then write an equation and solve for x.



One way to find the sum of the interior angles of the hexagon is to divide the figure into triangles. There are several different ways to do this, but keep in mind that we are trying to add the interior angles at the vertices. One way to divide the hexagon into triangles is to draw in all of the diagonals from a single vertex, as shown at right. Doing this forms four triangles, each with angle measures summing to 180°.

$$\underbrace{m\angle 1+m\angle 2+m\angle 3}_{180^\circ}+\underbrace{m\angle 4+m\angle 5+m\angle 6}_{180^\circ}+\underbrace{m\angle 7+m\angle 8+m\angle 9}_{180^\circ}+\underbrace{m\angle 10+m\angle 11+m\angle 12}_{180^\circ}$$
$$=4(180^\circ)=720^\circ$$

(Note: Students may have noticed that the number of triangles is always two less than the number of sides. This example illustrates why the sum of the interior angles of a polygon may be calculated using the formula $(n-2)180^\circ$, where n is the number of sides of the polygon.)

Now that we know what the sum of the angles is, we can write an equation, and solve for x.

$$(3x+1^\circ)+(4x+7^\circ)+(x+1^\circ)+(3x-5^\circ)+(5x-4^\circ)+(2x)=720^\circ$$
$$18x=720^\circ$$
$$x=40^\circ$$

Example 2

If the sum of the measures of the interior angles of a polygon is 2340°, how many sides does the polygon have?

Use the equation "sum of interior angles = $(n-2)180°$ " to write an equation and solve for n. The solution is shown at right.

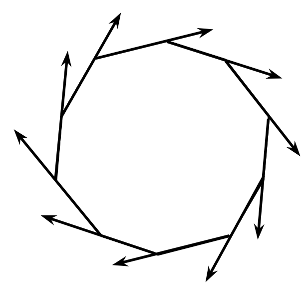
$$(n-2)\cdot 180° = 2340°$$
$$180°n - 360° = 2340°$$
$$180°n = 2700°$$
$$n = 15$$

Since $n = 15$, the polygon has 15 sides. It is important to note that if the answer is not a whole number, an error was made or there is no polygon with its interior angles summing to the measure given. Since the answer is the number of sides, the answer can only be a whole number. Polygons cannot have "7.2" sides!

Example 3

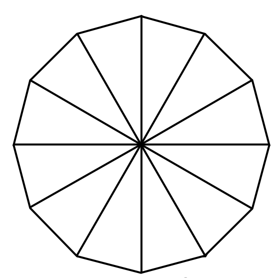
What is the measure of an exterior angle of a regular decagon?

A decagon is a 10-sided polygon. Since this figure is a regular decagon, all the angles and all the sides are congruent. The sum of the measures of the exterior angles of any polygon, one at each vertex, is always 360°, no matter how many sides the polygon has. In this case the exterior angles are congruent since the decagon is regular. The decagon at right has ten exterior angles drawn, one at each vertex. Therefore, each angle measures $\frac{360°}{10} = 36°$.



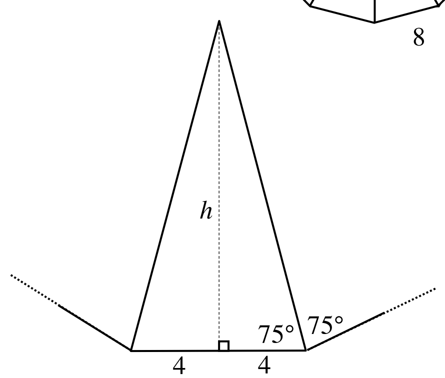
Example 4

A regular dodecagon (12 sided polygon) has a side length of 8 units. What is its area?



Solving this problem is going to require the use of several topics that have been studied. (Note: There is more than one way to solve this problem.) For this solution, we will imagine dividing the dodecagon into 12 congruent triangles, radiating from the center. If we find the area of one of them, then we can multiply it by 12 to get the area of the entire figure.

To focus on one triangle, copy and enlarge it. The triangle is isosceles, so drawing a segment from the vertex angle perpendicular to the base gives a height. This height also bisects the base (because this triangle is isosceles).



Since this is a dodecagon, we can find the sum of all the angles of the shape by using the formula:

$$(12-2)(180°) = 1800°$$

Since all the angles are congruent, each angle measures $1800° \div 12 = 150°$. The segments radiating from the center bisect each angle, so the base angle of the isosceles triangle is 75°. Now we can use trigonometry to find h.

$\tan 75° = \frac{h}{4}$

$h = 4 \tan 75°$

$h \approx 14.928$

Therefore the area of one of these triangles is: $A \approx \frac{1}{2}(8)(14.928) \approx 59.712$ square units

To find the area of the dodecagon, we multiply the area of one triangle by 12.

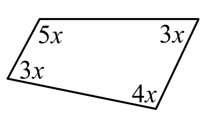
$$A \approx 12(59.712) \approx 716.544 \text{ square units}$$

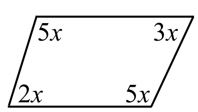
Problems

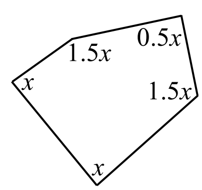
Find the measures of the angles in each problem below.

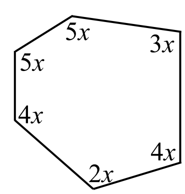
1. Find the sum of the interior angles in a 7-gon.
2. Find the sum of the interior angles in an 8-gon.
3. Find the size of each of the interior angle of a regular 12-gon.
4. Find the size of each of the interior angle of a regular 15-gon.
5. Find the size of each of the exterior angle of a regular 17-gon.
6. Find the size of each of the exterior angle of a regular 21-gon.

Solve for x in each of the figures below.

7. 

8. 

9. 

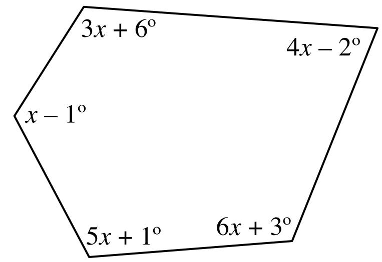
10. 

Complete each of the following problems.

11. Each exterior angle of a regular n-gon measures $16\frac{4}{11}°$. How many sides does this n-gon have?
12. Each exterior angle of a regular n-gon measures $13\frac{1}{3}°$. How many sides does this n-gon have?
13. Each angle of a regular n-gon measures 156°. How many sides does this n-gon have?
14. Each angle of a regular n-gon measures 165.6°. How many sides does this n-gon have?
15. Find the area of a regular pentagon with side length 8 cm.
16. Find the area of a regular hexagon with side length 10 ft.

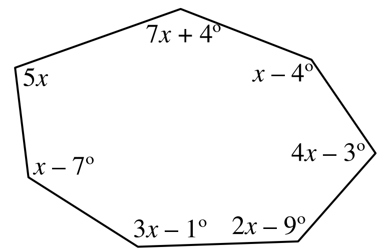
17. Find the area of a regular octagon with side length 12 m.

18. Find the area of a regular decagon with side length 14 in.



19. Using the pentagon at right, write an equation and solve for x.

20. Using the heptagon (7-gon) at right, write an equation and solve for x.



21. What is the sum of the measures of the interior angles of a 14-sided polygon?

22. What is the measure of each interior angle of a regular 16-sided polygon?

23. What is the sum of the measures of the exterior angles of a decagon (10-gon)?

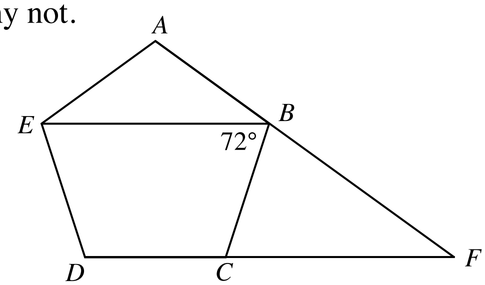
24. Each exterior angle of a regular polygon measures 22.5°. How many sides does the polygon have?

25. Does a polygon exist whose sum of the interior angles is 3060°? If so, how many sides does it have? If not, explain why not.

26. Does a polygon exist whose sum of the interior angles is 1350°? If so, how many sides does it have? If not, explain why not.

27. Does a polygon exist whose sum of the interior angles is 4410°? If so, how many sides does it have? If not, explain why not.

28. In the figure at right, *ABCDE* is a regular pentagon. Is $\overline{EB} \parallel \overline{DF}$? Justify your answer.



29. What is the area of a regular pentagon with a side length of 10 units?

30. What is the area of a regular 15-gon with a side length of 5 units?

Answers

1. 900°
2. 1080°
3. 150°
4. 156°
5. 21.1765°
6. 17.1429°
7. $x = 24^{\circ}$
8. $x = 30^{\circ}$
9. $x = 98.18^{\circ}$
10. $x = 31.30^{\circ}$
11. 22 sides
12. 27 sides
13. 15 sides
14. 25 sides
15. 110.1106 cm^2
16. 259.8076 ft^2
17. 695.2935 m^2
18. 1508.0649 in.^2
19. $19x + 7^{\circ} = 540^{\circ}, x \approx 28.05^{\circ}$
20. $23x - 20^{\circ} = 900^{\circ}, x = 40^{\circ}$
21. 2160°
22. 157.5°
23. 360°
24. 16 sides
25. 19 sides
26. No. The result is not a whole number.
27. No. The result is not a whole number.
29. Yes. Since *ABCDE* is a regular pentagon, the measure of each interior angle is 108°. Therefore, $m\angle DCB = 108^{\circ}$. Since $\angle DCB$ and $\angle FCB$ are supplementary, $m\angle FCB = 72^{\circ}$. The lines are parallel because the alternate interior angles are congruent.
30. ≈ 172.0 sq. units
31. ≈ 441.1 sq. units

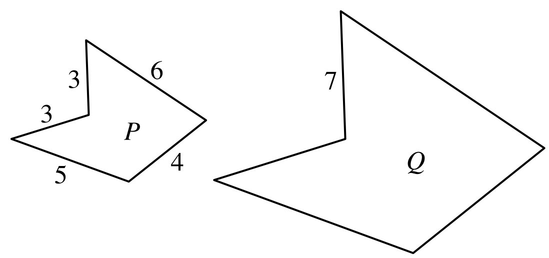
AREA RATIOS OF SIMILAR FIGURES 8.2.1 – 8.2.2

Students return to similarity once again to explore what happens to the area of a figure if it is reduced or enlarged. In Chapter 3, students learned about the ratio of similarity, also called the "zoom factor." If two similar figures have a ratio of similarity of $\frac{a}{b}$, then the ratio of their perimeters is also $\frac{a}{b}$, while the ratio of their areas is $\frac{a^2}{b^2}$.

See the Math Notes boxes in Lessons 8.2.1 and 9.1.5.

Example 1

The figures P and Q at right are similar.



a. What is the ratio of similarity?

b. What is the perimeter of figure P?

c. Use your previous two answers to find the perimeter of figure Q.

d. If the area of figure P is 34 square units, what is the area of figure Q?

The ratio of similarity is the ratio of the lengths of two corresponding sides. In this case, since we only have the length of one side of figure Q, we will use the side of P that corresponds to that side. Therefore, the ratio of similarity is $\frac{3}{7}$.

To find the perimeter of figure P, add up all the side lengths: 3 + 6 + 4 + 5 + 3 = 21. If the ratio of similarity of the two figures is $\frac{3}{7}$ then ratio of their perimeters is $\frac{3}{7}$ as well.

$$\frac{\text{perimeter } P}{\text{perimeter } Q} = \frac{3}{7}$$
$$\frac{21}{Q} = \frac{3}{7}$$
$$3Q = 147$$
$$\text{perimeter } Q = 49$$

If the ratio of similarity is $\frac{3}{7}$ then the ratio of the areas is $\left(\frac{3}{7}\right)^2 = \frac{9}{49}$.

$$\frac{\text{area } P}{\text{area } Q} = \left(\frac{3}{7}\right)^2$$
$$\frac{34}{Q} = \frac{9}{49}$$
$$9Q = 1666$$
$$\text{area } Q \approx 185.11 \text{ square units}$$

Example 2

Two rectangles are similar. If the area of the first rectangle is 49 square units, and the area of the second rectangle is 256 square units, what is the ratio of similarity between these two rectangles?

Since the rectangles are similar, if the ratio of similarity is $\frac{a}{b}$, then the ratio of their areas is $\frac{a^2}{b^2}$. We are given the areas so we know the ratio of their areas is $\frac{49}{256}$. Therefore we can write:

$$\frac{a^2}{b^2} = \frac{49}{256}$$

$$\frac{a}{b} = \sqrt{\frac{49}{256}} = \frac{\sqrt{49}}{\sqrt{256}} = \frac{7}{16}$$

The ratio of similarity between the two rectangles is $\frac{a}{b} = \frac{7}{16}$. This can be written as a decimal or as a fraction.

Problems

1. If figure A and figure B are similar with a ratio of similarity of $\frac{5}{4}$, and the perimeter of figure A is 18 units, what is the perimeter of figure B?

2. If figure A and figure B are similar with a ratio of similarity of $\frac{1}{8}$, and the area of figure A is 13 square units, what is the area of figure B?

3. If figure A and figure B are similar with a ratio of similarity of 6, that is, 6 to 1, and the perimeter of figure A is 54 units, what is the perimeter of figure B?

4. If figure A and figure B are similar and the ratio of their perimeters is $\frac{17}{6}$, what is their ratio of similarity?

5. If figure A and figure B are similar and the ratio of their areas is $\frac{32}{9}$, what is their ratio of similarity?

6. If figure A and figure B are similar and the ratio of their perimeters is $\frac{23}{11}$, does that mean the perimeter of figure A is 23 units and the perimeter of figure B is 11 units? Explain.

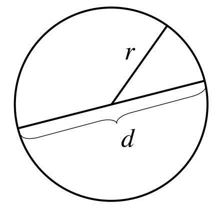
Answers

1. 14.4 units 2. 832 sq. units 3. 9 units 4. $\frac{17}{6}$ 5. $\frac{\sqrt{32}}{\sqrt{9}} \approx \frac{5.66}{3} \approx 1.89$

6. No, it just tells us the ratio. Figure A could have a perimeter of 46 units while figure B has a perimeter of 22 units.

CIRCUMFERENCE AND AREA OF CIRCLES 8.3.1 – 8.3.3

Students have found the area and perimeter of several polygons. Next they consider what happens to the area as more and more sides are added to a polygon. By exploring the area of a polygon with many sides, they learn that the limit of a polygon is a circle. They extend what they know about the perimeter and area of polygons to circles, and find the relationships for the circumference (C) and area (A) of circles.



$C = \pi d$ or $2\pi r$, $A = \pi r^2$

"C" is the circumference of the circle (a circle's perimeter), "d" is the diameter, and "r" is the radius. "π," which is in both formulas, is by definition the ratio $\frac{\text{circumference}}{\text{diameter}}$, and it is always a constant for any size circle.

Using these formulas, along with ratios, students are able to find the perimeter and area of shapes containing parts of circles.

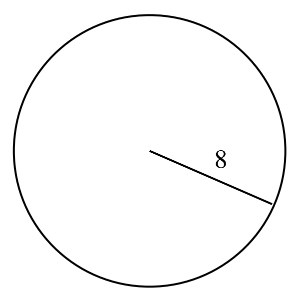
See the Math Notes boxes in Lessons 7.1.2, 8.3.2, and 8.3.3.

Example 1

The circle at right has a radius of 8 cm. What are the circumference and the area of the circle?

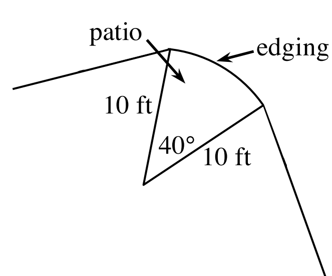
Using the formulas,

$$\begin{aligned} C &= 2\pi r \\ &= 2\pi(8) \\ &= 16\pi \\ &\approx 50.27 \text{ cm} \end{aligned} \qquad \begin{aligned} A &= \pi r^2 \\ &= \pi(8)^2 \\ &= 64\pi \\ &\approx 201.06 \text{ sq. cm} \end{aligned}$$



Example 2

Hermione has a small space on her corner lot that she would like to turn into a patio. To do this, she needs to do two things. First, she must know the length of the curved part, where she will put some decorative edging. Second, with the edging in place, she will need to purchase concrete to cover the patio. The concrete is sold in bags. Each bag will fill 2.5 square feet to the required depth of four inches. How much edging and concrete should Hermione buy?



The edging is a portion of the circumference of a circle with the center at point O and a radius of 10 feet. We can determine the exact fraction of the circle by looking at the measure of the central angle. Since the angle measures 40°, and there are 360° in the whole circle, this portion is $\frac{40°}{360°} = \frac{1}{9}$ of the circle. If we find the circumference and area of the whole circle, then we can take $\frac{1}{9}$ of each of those measurements to find the portion needed.

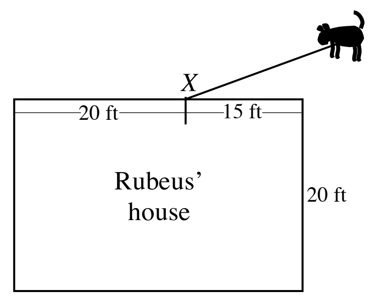
$$C = \tfrac{1}{9}(2\pi r) = \tfrac{1}{9}(2 \cdot \pi \cdot 10) = \tfrac{20\pi}{9} \approx 6.98 \text{ feet}$$

$$A = \tfrac{1}{9}\pi r^2 = \tfrac{1}{9} \cdot \pi \cdot (10)^2 = \tfrac{100\pi}{9} \approx 34.91 \text{ square feet}$$

Hermione should buy 7 feet of edging, and she should buy 14 bags of concrete (34.91 ÷ 2.5 ≈ 13.96 bags). Concrete is sold in full bags only.

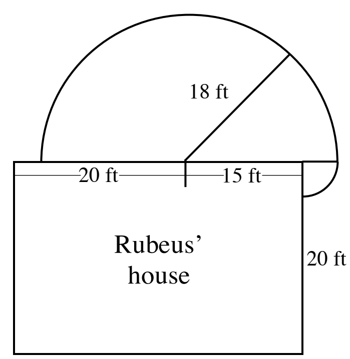
Example 3

Rubeus' dog Fluffy is tethered to the side of his house at point X. If Fluffy's rope is 18 feet long, how much area does Fluffy have to run in?



Because Fluffy is tethered to a point by a rope, he can only go where the rope can reach. Assuming that there are no obstacles, this area would be circular. Since Fluffy is blocked by the house, the area will only be a portion of a circle.

From point X, Fluffy can reach 18 feet to the left and right of point X. This initial piece is a semicircle. But, to the right of point X, the rope will bend around the corner of the house, adding a little more area for Fluffy. This smaller piece is a quarter of a circle with a radius of 3 feet.



Semicircle:

$$A = \tfrac{1}{2}\pi r^2 = \tfrac{18^2\pi}{2} = \tfrac{324\pi}{2} = 162\pi \approx 508.94$$

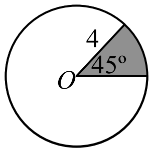
Quarter circle:

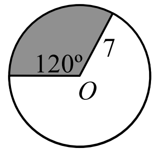
$$A = \tfrac{1}{4}\pi r^2 = \tfrac{3^2\pi}{4} = \tfrac{9\pi}{4} \approx 7.07$$

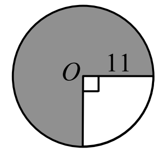
Fluffy has a total of 508.94 + 7.07 ≈ 516 square feet in which to run.

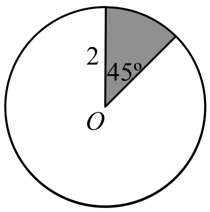
Problems

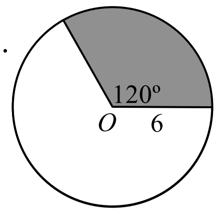
Calculate the area of the shaded sector in each circle below. Point O is the center of each circle.

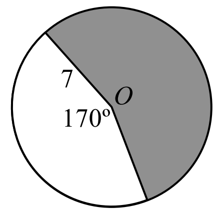
1. 

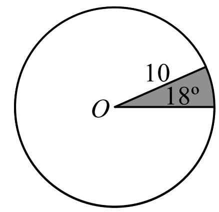
2. 

3. 

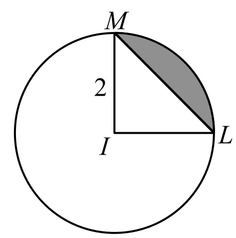
4. 

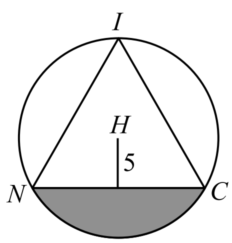
5. 

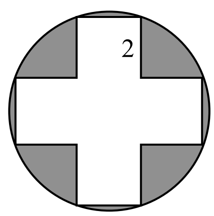
6. 

7. 

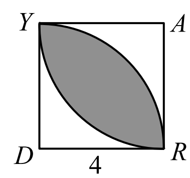
8. The shaded region in the figure is called a segment of the circle. It can be found by subtracting the area of ΔMIL from the sector MIL. Find the area of the segment of the circle.



9. 

10. 

11. *YARD* is a square; *A* and *D* are the centers of the arcs.



12. Find the area of a circular garden if the diameter of the garden is 30 feet.

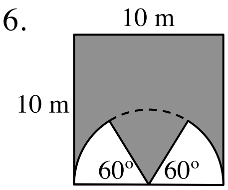
13. Find the area of a circle inscribed in a square whose diagonal is 8 feet long.

14. The area of a 60° sector of a circle is 10π m^2. Find the radius of the circle.

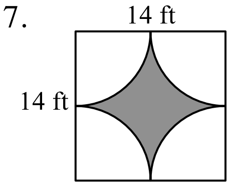
15. The area of a sector of a circle with a radius of 5 mm is 10π mm^2. Find the measure of its central angle.

Find the area of each shaded region.

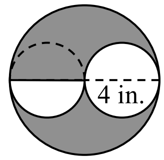
16.



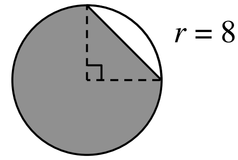
17.



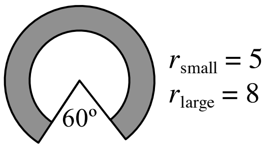
18.



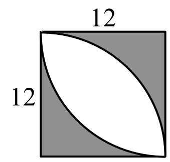
19.



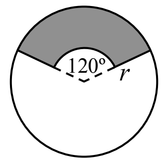
20.



21.



22. Find the length of the radius. The shaded area is 12π cm^2.



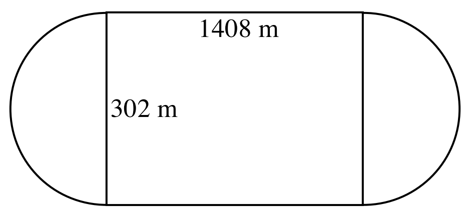
23. Find the arc length of the shaded sector in problem 1.

24. Find the arc length of the shaded sector in problem 2.

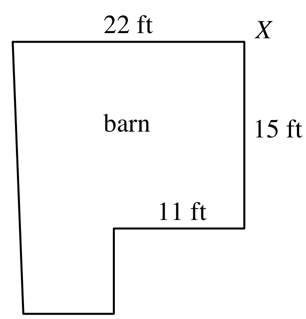
25. Find the arc length of the shaded sector in problem 3.

26. Find the arc length of the shaded sector in problem 4.

27. Kennedy and Tess are constructing a racetrack for their horses. The track encloses a field that is rectangular, with two semicircles at each end. A fence must surround this field. How much fencing will Kennedy and Tess need?



28. Rubeus has moved his dog Fluffy to a corner of his barn because he wants him to have more room to run. If Fluffy is tethered at point X on the barn with a 20 foot rope, how much area does Fluffy have to explore?



Answers

1. $2\pi \approx 6.28$ units^2
2. $\frac{49}{3}\pi \approx 51.31$ units^2
3. $\frac{363\pi}{4} \approx 285.10$ units^2
4. $\frac{\pi}{2}$ units^2
5. 12π units^2
6. $\frac{931\pi}{36}$ units^2
7. 5π units^2
8. $\pi - 2$ units^2
9. $\frac{100}{3}\pi - 25\sqrt{3}$ units^2
10. $10\pi - 20$ units^2
11. $8\pi - 16$ units^2
12. 225π ft^2
13. 8π ft^2
14. $2\sqrt{15}$ m
15. 144°
16. $100 - \frac{25}{3}\pi$ m^2
17. $196 - 49\pi$ ft^2
18. 10π in.^{2}
19. $48\pi + 32$ units^2
20. $\frac{65}{2}\pi$ units^2
21. ≈ 61.8 units^2
22. 6 cm
23. $\pi \approx 3.14$ units
24. $\frac{14\pi}{3} \approx 14.66$ units
25. $\frac{33\pi}{2} \approx 51.84$ units
26. $\frac{\pi}{2}$ units
27. $2816 + 302\pi \approx 3764.76$ meters of fencing
28. $200\pi + 100\pi + \frac{25\pi}{4} \approx 962.11$ square feet

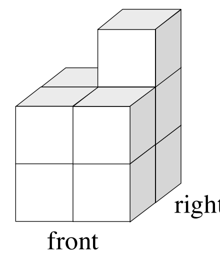
SOLIDS AND THEIR MEASUREMENTS 9.1.1 – 9.1.5

In this chapter, students examine three-dimensional shapes, known as solids. Students will work on visualizing these solids by building and then drawing them. Visualization is a useful, often overlooked skill in mathematics. By drawing solids students gain a better understanding of volume and surface area.

See the Math Notes boxes in Lessons 9.1.2, 9.1.3, and 9.1.5.

Example 1

The solid at right is built from individual cubes (blocks) stacked upon each other on a flat surface. (This means that no cubes are "floating.") Create a mat plan representing this solid. What is the volume of this solid?



This solid consists of stacked blocks. We are looking at the front, right side, and top of this solid. A mat plan shows a different perspective of a solid. It shows the footprint of the solid as well as how many blocks are in each stack. A mat plan is useful because, in the solid above, we cannot see the possible "hidden" blocks. A mat plan tells us exactly how many blocks are in the solid.

In this case, since we are creating the mat plan from the stacked blocks, there is more than one possible answer. If there are no hidden blocks, then the mat plan is the first diagram at right. If there is a hidden block, then the mat plan is the second one at right. It is helpful to visualize solids by building them with cubes. Build solids on a 3×5 card so that you can rotate the card to see the solid from all of its sides. Do this to make sure that one block is all that can be hidden in this drawing.

TWO POSSIBLE MAT PLANS

2	3
2	2

Front no hidden blocks

1	
2	3
2	2

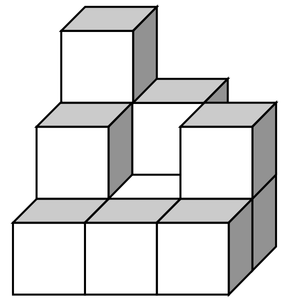
Front hidden block

The volume of this solid is the number of cubes it would take to build it. In this case, the volume is either 9 cubic units or 10 cubic units.

Example 2

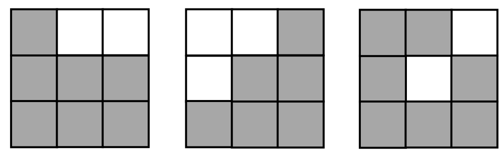
3	2	
2		2
1	1	1

At right is a mat plan of a solid. Build the solid. What is the volume of this solid? Draw the front, right, and top views, as well as the three-dimensional view of this solid.



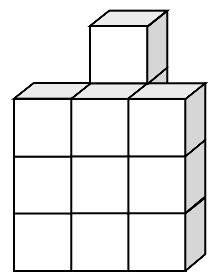
We find the volume by counting the number of blocks it would take to build this solid or by adding the numbers in the mat plan. The volume of this solid is 12 cubic units. To draw the different views of this solid, it is extremely helpful to build it out of cubes on a 3×5 card. Label the card with front, right, left, and back so that you can remember which side is which when rotating it. Remember that the standard three-dimensional view shows the top, right, and front views.

The individual views of each side are flat views. It is helpful to look at the solid at your eye level, so that only one side is visible at a time.



Front Right Top

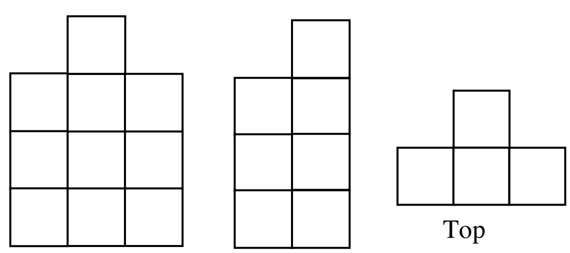
Example 3



If the figure at right is made with the fewest amount of cubes possible, what is its surface area?

The surface area is the sum of the areas of all the surfaces (or faces or sides) on the solid. If we build the solid on a card, then we can rotate the card and count the number of squares on each face, except the bottom. Note that because we never have floating cubes in our solids, the bottom has the same surface area as the top. If we draw every face of the solid, we can count the number of squares to find the surface area. Alternately, rather than drawing every face, we can draw only three views — the front, right, and top — and double those areas, because the back, left, and bottom are always their reflections with equivalent respective areas. Either way, we will arrive at the same answer.

From the front and back the solid looks the same and shows 10 squares.



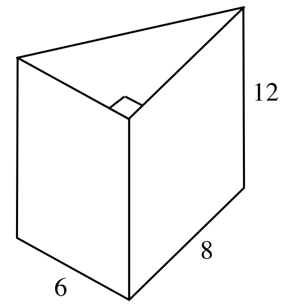
Front Right Top

The right and left views are reflections of each other and each shows seven squares.

The top and bottom views are also reflections of each other. They show four squares each.

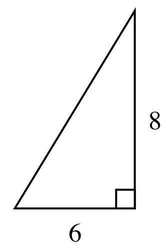
Therefore, the surface area is $10 + 10 + 7 + 7 + 4 + 4 = 42$ square units.

Example 4



The dimensions of the prism at right are shown. What are the volume and surface area of this prism?

A prism is a special type of polyhedron that has two congruent and parallel bases. In this problem, the bases are right triangles. The volume of a prism is found by multiplying the area of the base by the height of the prism. To understand this process, think of a prism as a stack of cubes. The base area tells you how many cubes are in one layer of the stack. The height tells you how many layers of cubes are in the figure.

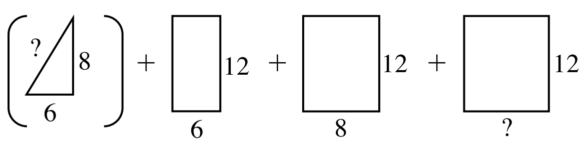


In this example, the base is a right triangle, so the area is $\frac{1}{2}bh$. Looking at the top of the prism might make it easier to find the area of the base represented by A_b. $A_b = \frac{1}{2}bh = \frac{1}{2}(6)(8) = 24$ square units, so there are 24 cubes in one layer.

To find the volume, we multiply this amount by the height, 12.

$$V = A_b h = (24)(12) = 288 \text{ cubic units}$$

To find the surface area of this prism, we will find the area of each of its faces, including the bases, and add the areas. One way to illustrate the sub-problems is to make sketches of the surfaces.

Surface Area = 2 

All of the surfaces are familiar shapes, namely, triangles and rectangles. We need to calculate the length of the rectangle on the back face (the last rectangle in the pictorial equation above). Fortunately, that length is also the hypotenuse of the right triangle of the base, so we can use the Pythagorean Theorem to find that length.

$$\begin{aligned} 6^2 + 8^2 &= ?^2 \\ 36 + 64 &= ?^2 \\ ?^2 &= 100 \\ ? &= \sqrt{100} = 10 \end{aligned}$$

Therefore the surface area is:
$$\begin{aligned} \text{S.A.} &= 2\left(\tfrac{1}{2}\cdot 6\cdot 8\right) + (6\cdot 12) + (8\cdot 12) + (10\cdot 12) \\ &= 48 + 72 + 96 + 120 \\ &= 336 \text{ square units} \end{aligned}$$

Example 5

The Styrofoam pieces used in packing boxes, known as "shipping peanuts," are sold in three box sizes: small, medium, and large. The small box has a volume of 1200 cubic inches. The dimensions on the "medium" box are twice the dimensions of the small box, and the "large" box has triple the dimensions of the small one. All three boxes are similar prisms. What are volumes of the medium and large boxes?

Since the boxes are similar, we can use the ratio of similarity to determine the volume of the medium and large boxes without knowing their actual dimensions. When figures are similar with ratio of similarity $\frac{a}{b}$, the ratio of the areas is $(\frac{a}{b})^2$ and the ratio of the volumes is $(\frac{a}{b})^3$. Since the medium box has dimensions twice the small box and the large box has dimensions three times the small box, we can write:

$$\frac{\text{medium box}}{\text{small box}} = \frac{2}{1} \qquad \frac{\text{large box}}{\text{small box}} = \frac{3}{1}$$

$$\frac{\text{volume of medium box}}{\text{volume of small box}} = \left(\frac{2}{1}\right)^3 \qquad \frac{\text{volume of large box}}{\text{volume of small box}} = \left(\frac{3}{1}\right)^3$$

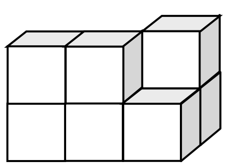
$$\frac{V_m}{1200} = \frac{8}{1} \qquad \frac{V_l}{1200} = \frac{27}{1}$$

Solving, $V_m = 8 \cdot 1200$ or $V_m = 9600$ cubic units and $V_l = 27 \cdot 1200$ or $V_l = 32{,}400$ cubic units.

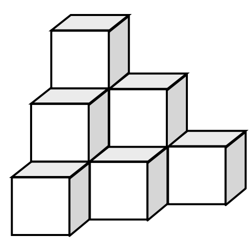
Problems

For each solid, calculate the volume and surface area, then draw a mat plan. Assume there are no hidden or floating cubes.

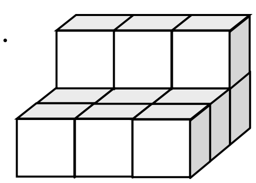
1.



2.



3.



For each mat plan, draw the solid, then calculate the volume and surface area.

4.

	2	2
5	4	1
4	1	1

5.

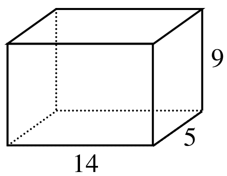
2	2	1
2		1
1	1	1

6.

3	3	2
	2	2
1	1	1

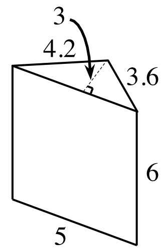
Calculate the volume and surface area of each prism.

7.

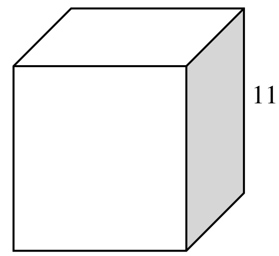


The base is a rectangle.

8.



9.

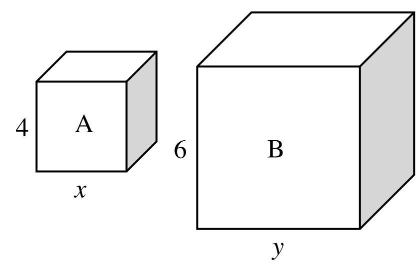


A cube.

10. At Cakes R Us, it is possible to buy round cakes in different sizes. The smallest cake has a diameter of 8 inches and a height of 4 inches, and requires 3 cups of batter. Another similar round cake has a diameter of 13 inches. How much batter would this cake require?

11. Prism A and prism B are similar with a ratio of similarity of 2:3. If the volume of prism A is 36 cubic units, what is the volume of prism B?

12. Two rectangular prisms are similar. The smaller, A, has a height of 4 units while the larger, B, has a height of 6 units.



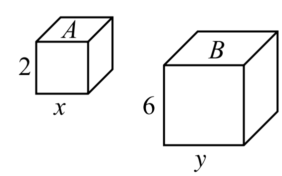
a. What is the scale factor from prism A to prism B?

b. What is the ratio of the lengths of the edges labeled x and y?

c. What is the ratio of their surface areas? What is the ratio of their volumes?

d. A third prism C is similar to prisms A and B. Prism C's height is 10 units. If the volume of prism A is 24 cubic units, what is the volume of prism C?

13. Two rectangular prisms are similar. The smaller, A, has a height of 4 units while the larger, B, has a height of 6 units.



a. What is the scale factor from prism B to prism A?

b. What would be the ratio of the lengths of the edges labeled x and y?

c. What is the ratio of their surface areas? What is the ratio of their volumes?

d. A third prism, C is similar to prisms A and B. Prism C's height is 10 units. If the volume of prism A is 20 cubic units, what is the volume of prism C?

14. If rectangle A and rectangle B have a ratio of similarity of 5:4, what is the area of rectangle B if the area of rectangle A is 24 square units?

15. If rectangle *A* and rectangle *B* have a ratio of similarity of 2:3, what is the area of rectangle *B* if the area of rectangle *A* is 46 square units?

16. If rectangle *A* and rectangle *B* have a ratio of similarity of 3:4, what is the area of rectangle *B* if the area of rectangle *A* is 82 square units?

17. If rectangle *A* and rectangle *B* have a ratio of similarity of 1:5, what is the area of rectangle *B* if the area of rectangle *A* is 24 square units?

18. Rectangle *A* is similar to rectangle *B*. The area of rectangle *A* is 81 square units while the area of rectangle *B* is 49 square units. What is the ratio of similarity between the two rectangles?

19. Rectangle *A* is similar to rectangle *B*. The area of rectangle *B* is 18 square units while the area of rectangle *A* is 12.5 square units. What is the ratio of similarity between the two rectangles?

20. Rectangle *A* is similar to rectangle *B*. The area of rectangle *A* is 16 square units while the area of rectangle *B* is 100 square units. If the perimeter of rectangle *A* is 12 units, what is the perimeter of rectangle *B*?

21. If prism *A* and prism *B* have a ratio of similarity of 2:3, what is the volume of prism *B* if the volume of prism *A* is 36 cubic units?

22. If prism *A* and prism *B* have a ratio of similarity of 1:4, what is the volume of prism *B* if the volume of prism *A* is 83 cubic units?

23. If prism *A* and prism *B* have a ratio of similarity of 6:11, what is the volume of prism *B* if the volume of prism *A* is 96 cubic units?

24. Prism *A* and prism *B* are similar. The volume of prism *A* is 72 cubic units while the volume of prism *B* is 1125 cubic units. What is the ratio of similarity between these two prisms?

25. Prism *A* and prism *B* are similar. The volume of prism *A* is 27 cubic units while the volume of prism *B* is approximately 512 cubic units. If the surface area of prism *B* is 128 square units, what is the surface area of prism *A*?

26. The corresponding diagonals of two similar trapezoids are in the ratio of 1:7. What is the ratio of their areas?

27. The ratio of the perimeters of two similar parallelograms is 3:7. What is the ratio of their areas?

28. The ratio of the areas of two similar trapezoids is 1:9. What is the ratio of their altitudes?

29. The areas of two circles are in the ratio of 25:16. What is the ratio of their radii?

30. The ratio of the volumes of two similar circular cylinders is 27:64. What is the ratio of the diameters of their similar bases?

31. The surface areas of two cubes are in the ratio of 49:81. What is the ratio of their volumes?

32. The ratio of the weights of two spherical steel balls is 8:27. What is the ratio of the diameters of the two steel balls?

Answers

1.

		2
2	2	1

V = 7 cu. units

SA = 28 sq. units

2.

3	2	1
2	1	
1		

V = 10 cu. units

SA = 36 sq. units

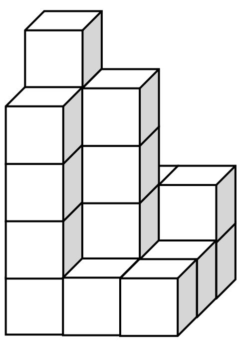
3.

2	2	2
1	1	1
1	1	1

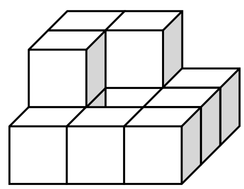
V = 12 cu. units

SA = 38 sq. units

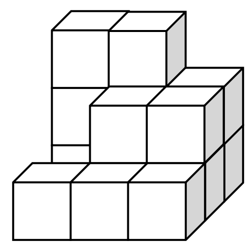
4. V = 20 cu. units
SA = 60 sq. units



5. V = 11 cu. units
SA = 36 sq. units



6. V = 15 cu. units
SA = 46 sq. units



7. V = 630 cu. units
SA = 482 sq. units

8. V = 45 cu. units
SA = 91.8 sq. units

9. V = 1331 cu. units
SA = 726 sq. units

10. ≈ 12.87 cups

11. 121.5 cu. units

12. a. $\frac{4}{6} = \frac{2}{3}$ b. $\frac{x}{y} = \frac{4}{6} = \frac{2}{3}$ c. $\frac{16}{36} = \frac{4}{9}, \frac{64}{216} = \frac{8}{27}$ d. 375 cu. units

13. a. $\frac{6}{2} = \frac{3}{1}$ b. $\frac{x}{y} = \frac{2}{6} = \frac{1}{3}$ c. $\frac{4}{36} = \frac{1}{9}, \frac{8}{216} = \frac{1}{27}$ d. 2500 cu. units

14. 15.36 units^2

15. 103.5 units^2

16. ≈ 145.8 units^2

17. 600 units^2

18. $\frac{9}{7}$

19. $\frac{6}{5}$

20. 30 units

21. 121.5

22. 5312

23. ≈ 591.6

24. $\frac{2}{5}$

25. ≈ 18 units^2

26. $\frac{1}{49}$

27. $\frac{9}{49}$

28. $\frac{1}{3}$

29. $\frac{5}{4}$

30. $\frac{3}{4}$

31. $\frac{343}{729}$

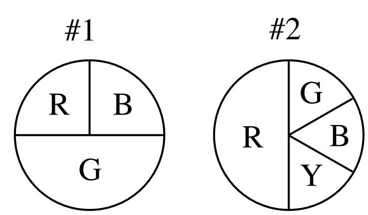
32. $\frac{2}{3}$

CONDITIONAL PROBABILITY AND TWO-WAY TABLES 10.2.1 – 10.2.3

The probability of one event occurring, knowing that a second event has already occurred is called a conditional probability. Two-way tables are useful to visualize conditional probability situations.

See the Math Notes boxes in Lessons 10.2.1 and 10.2.3.

Example 1



For the spinners at right, assume that the smaller sections of spinner #1 are half the size of the larger section and for spinner #2 assume that the smaller sections are one third the size of the larger section.

a. Draw a diagram for spinning twice.

b. What is the probability of getting the same color twice?

c. If you know you got the same color twice, what is the probability it was red?

The diagram for part (a) is shown at right. Note that the boxes do not need to be to scale. The circled boxes indicate getting the same color and the total probability for part (b) is: $\frac{1}{8}+\frac{1}{12}+\frac{1}{24}=\frac{6}{24}=\frac{1}{4}$.

	$R(\frac{1}{2})$	$G(\frac{1}{6})$	$B(\frac{1}{6})$	$Y(\frac{1}{6})$
$R(\frac{1}{4})$	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$
$B(\frac{1}{4})$	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$
$G(\frac{1}{2})$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

For part (c), both red is $\frac{1}{8}$ out of the $\frac{1}{4}$ from part (b) and so the probability the spinner was red knowing that you got the same color twice is $\frac{1}{8}\div\frac{1}{4}=\frac{1}{8}\cdot\frac{4}{1}=\frac{1}{2}$.

Example 2

A soda company conducted a taste test for three different kinds of soda that it makes.
It surveyed 200 people in each age group about their favorite flavor and the results are shown in the table below.

Age	Soda A	Soda B	Soda C
Under 20	30	44	126
20 to 39	67	75	58
40 to 59	88	78	34
60 and over	141	49	10

a. What is the probability that a participant chose Soda C or was under 20 years old?

b. What is the probability that Soda A was chosen?

c. If Soda A was chosen, what is the probability that the participant was 60 years old or older?

For part (a), using the addition rule:
$P(C \text{ or } <20) = P(C) + P(<20) - P(C \text{ and } <20) = \frac{228}{800} + \frac{200}{800} - \frac{126}{800} = \frac{302}{800} = 0.3775$.

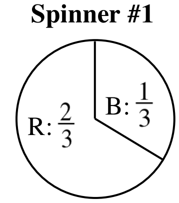
For part (b), adding the participants selecting Soda A: $\frac{30+67+88+141}{800} = \frac{326}{800} = 0.4075$.

For part (c), taking only the participants over 60 selecting Soda A out of all those selecting Soda A: $\frac{141}{30+67+88+141} \approx 0.43$.

Problems

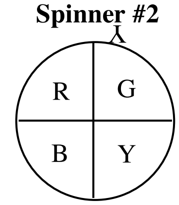
1. Two normal dice are thrown.
   a. How many ways are there to get 7 points?
   b. What is the probability of getting 7 points?
   c. If you got 7 points, what is the probability that one die was a 5?

Spinner #1



2. Elizabeth and Scott are playing game at the state fair that uses two spinners which are shown in the diagrams at right. The player spins both wheels and if the colors match you win a prize.
   a. Make a probability diagram for this situation.
   b. What is the probability of winning a prize?
   c. If you won a prize, what is the probability that the matching colors were red?

Spinner #2



3. The probability that it is Friday and a Sarah is absent is $\frac{1}{20}$. Since the school week has 5 days, the probability it is Friday is $\frac{1}{5}$. If today is Friday, what is the probability that Sarah is absent?

4. An airline wants to determine if passengers not checking luggage is related to people being on business trips. Data for 1000 random passengers at an airport was collected and summarized in the table below.

	Checked Baggage	No Checked Baggage
Traveling for business	103	387
Not traveling for business	216	294

   a. What is the probability of traveling and not checking baggage?

   b. If the passenger is traveling for business, what is the probability of not having checked baggage?

5. In Canada, 92% of the households have televisions. 72% of households have televisions and Internet access. What is the probability that a house has Internet given that it has a television?

6. There is a 25% chance that Claire will have to work tonight and cannot study for the big math test. If Claire studies, then she has an 80% chance of earning a good grade. If she does not study, she only have a 30% chance of earning a good grade.

   a. Draw a diagram to represent this situation.

   b. Calculate the probability of Claire earning a good grade on the math test.

   c. If Claire earned a good grade, what is the probability that she studied?

7. A bag contains 4 blue marbles and 2 yellow marbles. Two marbles are randomly chosen (the first marble is NOT replaced before drawing the second one).

   a. What is the probability that both marbles are blue?

   b. What is the probability that both marbles are yellow?

   c. What is the probability of one blue and then one yellow? If you are told that both selected marbles are the same color, what is the probability that both are blue?

8. At Cal's Computer Warehouse, Cal wants to know the probability that a customer who comes into his store will buy a computer or a printer. He collected the following data during a recent week: 233 customers entered the store, 126 purchased computers, 44 purchased printers, and 93 made no purchase.

   a. Draw a Venn diagram to represent the situation.

   b. From this data, what is the probability that the next customer who comes into the store will buy a computer or a printer?

   c. Cal has promised a raise for his salespeople if they can increase the probability that the customers who buy computers also buy printers. For the given data, what is the probability that if a customer bought a computer, he or she also bought a printer?

9. A survey of 200 recent high school graduates found that 170 had driver licenses and 108 had jobs. Twenty-one graduates said that they had neither a driver license nor a job.

   a. Draw a two-way table to represent the situation.

   b. If one of these 200 graduates was randomly selected, what is the probability that he or she has a job and no license?

   c. If the randomly selected graduate is known to have a job, what is the probability that he or she has a license?

10. At McDougal's Giant Hotdogs 15% of the workers are under 18 years old. The most desirable shift is 4-8pm and 80% of the workers under 18 years old have that shift. 30% of the 18 year old or over workers have the 4-8pm shift.

   a. Represent these probabilities in a two-way table.

   b. What is the probability that a randomly selected worker is 18 or over and does not work the 4-8pm shift?

   c. What is the probability that a randomly selected worker from the 4-8pm shift is under 18 years old?

Answers

1. a. 6 ways b. $\frac{6}{36} = \frac{1}{6}$ c. $\frac{2}{36} \div \frac{6}{36} = \frac{1}{3}$

2. a. See diagram at right.
   b. $\frac{1}{6} + \frac{1}{12} = \frac{1}{4}$
   c. $\frac{1}{6} \div \frac{1}{4} = \frac{2}{3}$

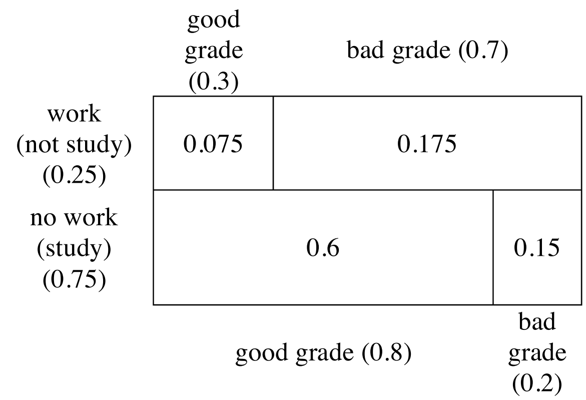
	$R(\frac{1}{4})$	$B(\frac{1}{4})$	$G(\frac{1}{4})$	$Y(\frac{1}{4})$
$R(\frac{2}{3})$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$B(\frac{1}{3})$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

3. $\frac{1}{20} \div \frac{1}{5} = \frac{1}{4}$

4. a. $\frac{681}{1000} = 0.681$ b. $\frac{387}{103+387} \approx 0.79$

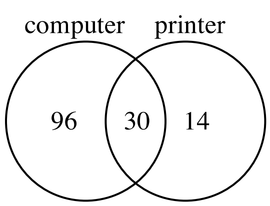
5. $\frac{72\%}{92\%} \approx 78\%$

6. a. See diagram at right.
   b. $0.075 + 0.60 = 0.675 = 67.5\%$
   c. $\frac{0.60}{0.675} \approx 0.89$



7. a. $\frac{4}{6} \cdot \frac{3}{5} = \frac{2}{5}$
   b. $\frac{2}{6} \cdot \frac{1}{5} = \frac{1}{15}$
   c. $\frac{4}{6} \cdot \frac{2}{5} = \frac{4}{15}$
   d. $\frac{2}{5} \div (\frac{2}{5} + \frac{1}{15}) = \frac{6}{7}$

8. a.



   b. $\frac{140}{233}$ c. $\frac{30}{126}$

9. a. See diagram at right.
   b. $\frac{9}{200}$
   c. $\frac{99}{108}$

	Job	No Job	
License	99	71	170
No Lic.	9	21	30
	108	92	200

10. a. See diagram at right.
    b. 0.595
    c. $\frac{0.12}{0.375} \approx 0.32$

	4–8pm	other	
< 18	$(0.15)(0.80) = 0.12$	0.03	0.15
≥ 18	$(0.3)(0.85) = 0.255$	0.595	0.85
	0.375	0.625	1

SOLIDS 11.1.1 – 11.1.5

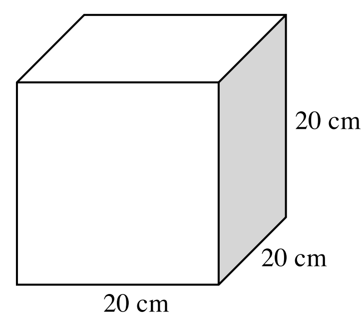
Students have already worked with solids, finding the volume and surface area of prisms and other shapes built with blocks. Now these skills are extended to finding the volumes and surface areas of pyramids, cones, and spheres.

See the Math Notes boxes in Lessons 11.1.2, 11.1.3, 11.1.4, 11.1.5, and 11.2.2.

Example 1

A regular hexahedron has an edge length of 20 cm. What are the surface area and volume of this solid?

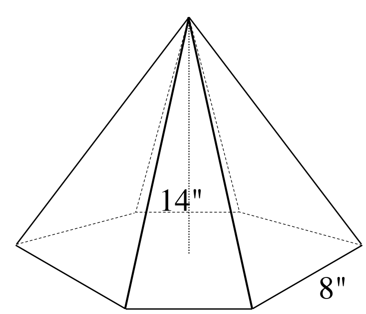
Although the name "regular hexahedron" might sound intimidating, it just refers to a regular solid with six (hexa) faces. As defined earlier, regular means all angles are congruent and all side lengths are congruent. A regular hexahedron is just a cube, so all six faces are congruent squares.



To find the volume of the cube, we can use our previous knowledge: multiply the area of the base by the height. Since the base is a square, its area is 400 square cm. The height is 20 cm, therefore the volume is (400)(20) = 8000 cubic cm.

To calculate the surface area we will find the sum of the areas of all six faces. Since each face is a square and they are all congruent, this will be fairly easy. The area of one square is 400 square cm, and there are six of them. Therefore the surface area is 2400 square cm.

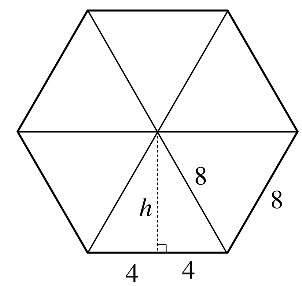
Example 2



The base of the pyramid at right is a regular hexagon. Using the measurements provided, calculate the surface area and volume of the pyramid.

The volume of any pyramid is $V = \frac{1}{3} A_b h$ (h is the height of the pyramid and A_b is the area of the base). We calculate the surface area the same way we do for all solids: find the area of each face and base, then add them all together. The lateral faces of the pyramid are all congruent triangles. The base is a regular hexagon. Since we need the area of the hexagon for both the volume and the surface area, we will find it first.

There are several ways to find the area of a regular hexagon. One way is to cut the hexagon into six congruent equilateral triangles, each with a side of 8". If we can find the area of one triangle, then we can multiply by 6 to find the area of the hexagon. To find the area of one triangle we need to find the value of h, the height of the triangle. Recall that we studied these triangles earlier; remember that the height cuts the equilateral triangle into two congruent 30°-60°-90° triangles. To find h, we can use the Pythagorean Theorem, or if you remember the pattern for a 30°-60°-90° triangle, we can use that. With either method we find that $h = 4\sqrt{3}$ ". Therefore the area of one equilateral triangle is shown at right.

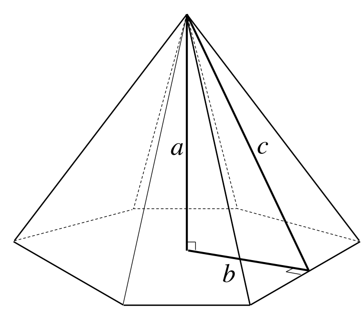


$$\begin{aligned} A &= \tfrac{1}{2}bh \\ &= \tfrac{1}{2}\cdot(8)\cdot\left(4\sqrt{3}\right) \\ &= 16\sqrt{3} \approx 27.71 \text{ in.}^2 \end{aligned}$$

The area of the hexagon is $6\cdot 16\sqrt{3} = 96\sqrt{3} \approx 166.28 \text{ in.}^2$. Now find the volume of the pyramid using the formula as shown at right.

$$\begin{aligned} V &= \tfrac{1}{3}A_b h \\ &= \tfrac{1}{3}\cdot\left(96\sqrt{3}\right)\cdot(14) \\ &= 448\sqrt{3} \approx 776 \text{ in.}^3 \end{aligned}$$

Next we need to find the area of one of the triangular faces. These triangles are slanted, and the height of one of them is called a slant height. The problem does not give us the value of the slant height (labeled c at right), but we can calculate it based on the information we already have.



A cross section of the pyramid at right shows a right triangle in its interior. One leg is labeled a, another b, and the hypotenuse c. The original picture gives us a = 14". The length of b we found previously: it is the height of one of the equilateral triangles in the hexagonal base. Therefore, $b = 4\sqrt{3}$ ". To calculate c, we use the Pythagorean Theorem.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 14^2 + (4\sqrt{3})^2 &= c^2 \\ 196 + 48 &= c^2 \\ c^2 &= 244 \\ c = \sqrt{244} &= 2\sqrt{61} \approx 15.62\text{"} \end{aligned}$$

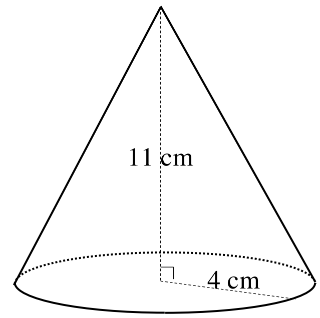
The base of one of the slanted triangles is 8", the length of the side of the hexagon. Therefore the area of one slanted triangle is $8\sqrt{61} \approx 62.48 \text{ in.}^2$ as shown below right.

$$\begin{aligned} A &= \tfrac{1}{2}\cdot b\cdot h \\ &= \tfrac{1}{2}\cdot(8)\cdot\left(2\sqrt{61}\right) \\ &= 8\sqrt{61} \approx 62.48 \text{ in.}^2 \end{aligned}$$

Since there are six of these triangles, the area of the lateral faces is $6(8\sqrt{61}) = 48\sqrt{61} \approx 374.89 \text{ in.}^2$.

Now we have all we need to find the total surface area: $96\sqrt{3} + 48\sqrt{61} \approx 541.17 \text{ in.}^2$.

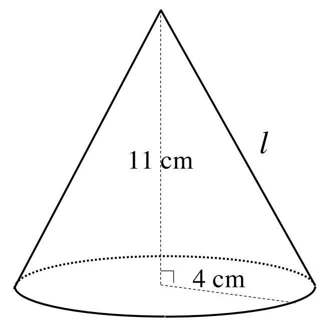
Example 3



The cone at right has the measurements shown. What are the lateral surface area and volume of the cone?

The volume of a cone is the same as the volume of any pyramid: $V=\frac{1}{3}A_bh$. The only difference is that the base is a circle, but since we know how to find the area of a circle ($A=\pi r^2$), we find the volume as shown at right.

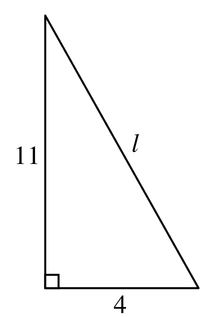
$$\begin{aligned} V &= \tfrac{1}{3}A_bh \\ &= \tfrac{1}{3}\left(\pi r^2\right)h \\ &= \tfrac{1}{3}\left(\pi\cdot 4^2\right)\cdot 11 \\ &= \tfrac{1}{3}\cdot(176\pi)=\tfrac{176\pi}{3} \\ &\approx 184.3\text{ cm}^3 \end{aligned}$$



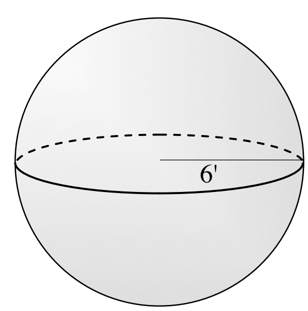
Calculating the lateral surface area of a cone is a different matter. If we think of a cone as a child's party hat, we can imagine cutting it apart to make it lay flat. If we did, we would find that the cone is really a sector of a circle – not the circle that makes up the base of the cone, but a circle whose radius is the slant height of the cone. By using ratios we can come up with the formula for the lateral surface area of the cone, SA = πrl, where r is the radius of the base and l is the slant height. In this problem, we have r, but we do not have l. Find it by taking a cross section of the cone to create a right triangle. The legs of the right triangle are 11 cm and 4 cm, and l is the hypotenuse. Using the Pythagorean Theorem we can calculate $l\approx 11.7$ cm, as shown below right.

$$\begin{aligned} 4^2+11^2 &= l^2 \\ 16+121 &= l^2 \\ l^2 &= 137 \\ l &= \sqrt{137}\approx 11.7\text{ cm} \end{aligned}$$

Now we can calculate the lateral surface area:
SA = $\pi(4)(11.7)\approx 147.1$ cm^2



Example 4

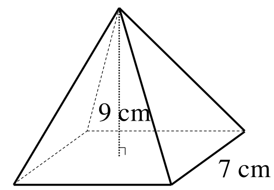


The sphere at right has a radius of 6 feet. Calculate the surface area and the volume of the sphere.

Since spheres are related to circles, we should expect that the formulas for the surface area and volume will have π in them. The surface area of a sphere with radius r is $4\pi r^2$. Since we know the radius of the sphere is 6, SA = $4\pi(6)^2=144\pi\approx 452.39$ ft^2. To find the volume of the sphere, we use the formula V = $\frac{4}{3}\pi r^3$. Therefore, V = $\frac{4}{3}\pi(6)^3=\frac{4\cdot 216\cdot\pi}{3}=288\pi\approx 904.78$ ft^3.

Problems

1. The figure at right is a square based pyramid. Calculate its surface area and its volume.

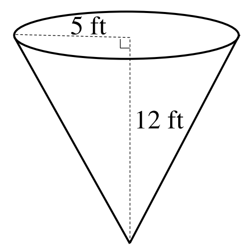


2. Another pyramid, congruent to the one in the previous problem, is glued to the bottom of the first pyramid, so that their bases coincide. What is the name of the new solid? Calculate the surface area and volume of the new solid.

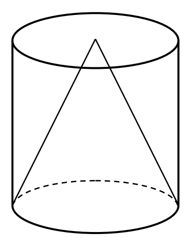
3. A regular pentagon has a side length of 10 in. Calculate the area of the pentagon.

4. The pentagon of the previous problem is the base of a right pyramid with a height of 18 in. What is the surface area and volume of the pyramid?

5. What is the total surface area and volume of the cone at right?

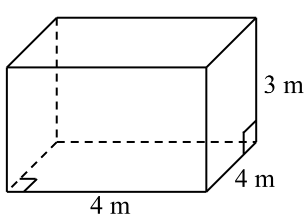


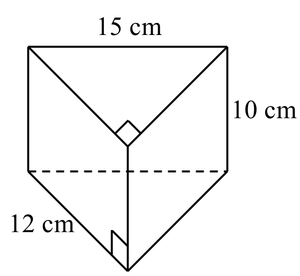
6. A cone fits perfectly inside a cylinder as shown. If the volume of the cylinder is 81π cubic units, what is the volume of the cone?

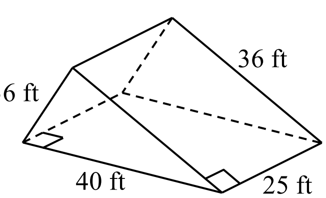


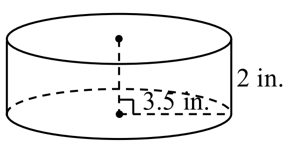
7. A sphere has a radius of 12 cm. What are the surface area and volume of the sphere?

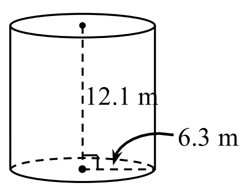
Find the volume of each figure.

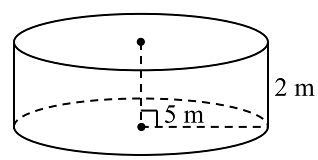
8. 

9. 

10. 

11. 

12. 

13. 

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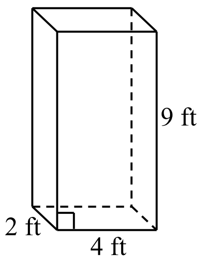

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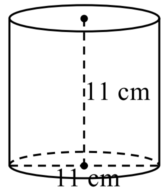

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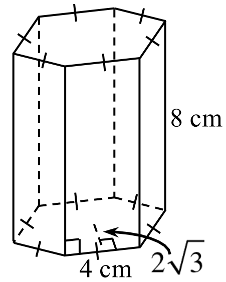

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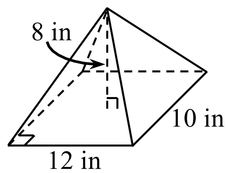

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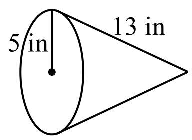

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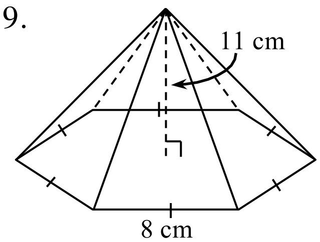

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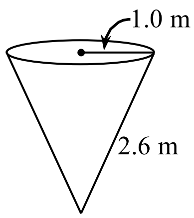

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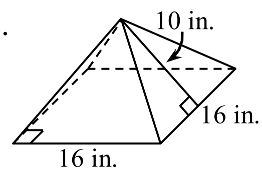

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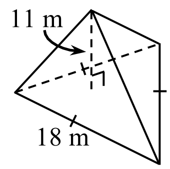

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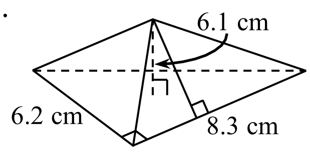

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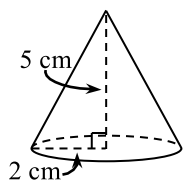

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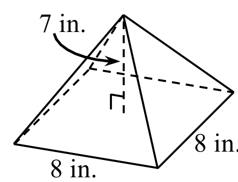
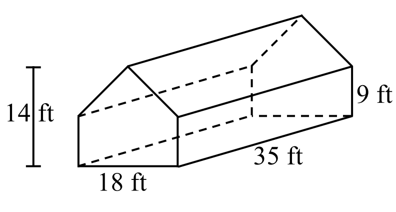
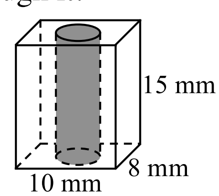


26. Find the volume of the solid shown.



27. Find the volume of the remaining solid after a hole with a diameter of 4 mm is drilled through it.



Find the total surface area of the figures in the previous volume problems.

28. Problem 8
29. Problem 9
30. Problem 10
31. Problem 12
32. Problem 13
33. Problem 14
34. Problem 15
35. Problem 16
36. Problem 17
37. Problem 21
38. Problem 25
39. Problem 26

Use the given information to find the volume of the cone.

40. radius = 1.5 in.
height = 4 in.

41. diameter = 6 cm
height = 5 cm

42. base area = 25π
height = 3

43. base circum. = 12π
height = 10

44. diameter = 12
slant height = 10

45. lateral area = 12π
radius = 1.5

Use the given information to find the lateral area of the cone.

46. radius = 8 in.
slant height = 1.75 in.

47. slant height = 10 cm
height = 8 cm

48. base area = 25π
slant height = 6

49. radius = 8 cm
height = 15 cm

50. volume = 100π
height = 5

51. volume = 36π
radius = 3

Use the given information to find the volume of the sphere.

52. radius = 10 cm

53. diameter = 10 cm

54. circumference of
great circle = 12π

55. surface area = 256π

56. circumference of
great circle = 20 cm

57. surface area = 100

Use the given information to find the surface area of the sphere.

58. radius = 5 in.

59. diameter = 12 in.

60. circumference of
great circle = 14

61. volume = 250

62. circumference of
great circle = π

63. volume = $\frac{9\pi}{2}$

Answers

1. $V = 147\text{ cm}^3$, $SA \approx 184.19\text{ cm}^2$
2. Octahedron, $V = 294\text{ cm}^3$, $SA \approx 270.38\text{ cm}^2$
3. $A \approx 172.05\text{ in.}^2$
4. $V = 1032.29\text{ in.}^3$, $SA \approx 653.75\text{ in.}^2$
5. $V \approx 314.16\text{ ft}^3$, $SA = 90\pi \approx 282.74\text{ ft}^2$
6. 27π cubic units
7. $SA = 576\pi \approx 1089.56\text{ cm}^2$, $V = 2304\pi \approx 7238.23\text{ cm}^3$
8. 48 m^3
9. 540 cm^3
10. 14966.6 ft^3
11. 76.9 in.^3
12. 1508.75 m^3
13. 157 m^3
14. 72 ft^3
15. 1045.4 cm^3
16. 332.6 cm^3
17. 320 in.^3
18. 314.2 in.^3
19. 609.7 cm^3
20. 2.5 m^3
21. 512 m^3
22. 514.4 m^3
23. 2.3 cm^3
24. 20.9 cm^3
25. 149.3 in.^3
26. 7245 ft^3
27. 1011.6 mm^3
28. 80 m^2
29. 468 cm^2
30. 3997.33 ft^2
31. 727.98 m^2
32. $50\pi + 20\pi \approx 219.8\text{ m}^2$
33. 124 ft^2
34. $121\pi + 189.97 \approx 569.91\text{ cm}^2$
35. $192 + 48\sqrt{3} \approx 275.14\text{ cm}^2$
36. 213.21 in.^2
37. 576 in.^2
38. 193.0 in.^2
39. 2394.69 ft^2
40. $3\pi \approx 9.42\text{ in.}^3$
41. $15\pi \approx 47.12\text{ cm}^3$
42. $25\pi \approx 78.54\text{ units}^3$
43. $120\pi \approx 376.99\text{ units}^3$
44. $96\pi \approx 301.59\text{ units}^3$
45. $\approx 18.51\text{ units}^3$
46. $14\pi \approx 43.98\text{ in.}^2$
47. $60\pi \approx 188.50\text{ cm}^2$
48. $30\pi \approx 94.25\text{ units}^2$
49. $136\pi \approx 427.26\text{ cm}^2$
50. $\approx 224.35\text{ units}^2$
51. 116.58 units^2
52. $\frac{4000\pi}{3} \approx 4188.79\text{ cm}^3$
53. $\frac{500\pi}{3} \approx 523.60\text{ cm}^3$
54. $288\pi \approx 904.79\text{ units}^3$
55. $\frac{2048\pi}{3} \approx 2144.66\text{ units}^3$
56. $\approx 135.09\text{ units}^3$
57. $\approx 94.03\text{ units}^3$
58. $100\pi \approx 314.16\text{ units}^2$
59. $144\pi \approx 452.39\text{ untis}^2$
60. $\approx 62.39\text{ units}^2$
61. $\approx 191.91\text{ units}^2$
62. $\pi \approx 3.14\text{ units}^2$
63. $9\pi \approx 28.27\text{ units}^2$

PRINCIPLES OF COUNTING 10.3.1 –10.3.5

Students take on challenging problems using the Fundamental Principle of Counting, permutations, and combinations to compute probabilities. These techniques are essential when the sample space is too large to model or to count.

See the Math Notes boxes in Lessons 10.3.1, 10.3.2, 10.3.3, and 10.3.5.

Example 1

Twenty-three people have entered the pie-eating contest at the county fair. The first place pie-eater (the person eating the most pies in fifteen minutes) wins a pie each week for a year. Second place will receive new baking ware to make his/her own pies, and third place will receive the *Sky High Pies* recipe book. How many different possible top finishers are there?

Since the prizes are different for first, second, and third place, the order of the top finishers matters. We can use a decision chart to determine the number of ways we can have winners. How many different people can come in first? Twenty-three. Once first place is "chosen" (i.e., removed from the list of contenders) how many people are left to take second place? Twenty-two. This leaves twenty-one possible third place finishers. Just as with the branches on the tree diagram, multiply these numbers to determine the number of arrangements: (23)(22)(21) = 10,626.

23	22	21
First	Second	Third

Example 2

Fifteen students are participating in a photo-shoot for a layout in the new journal *Mathmaticious*. In how many ways can you arrange:

a. Eight of them? b. Two of them? c. Fifteen of them?

We can use a decision chart for each of these situations, but there is another, more efficient method for answering these questions. An arrangement of items where order matters is called a **permutation**, and in this case, since changing the order of the students changes the layout, the order matters.

With a permutation, you need to know the total number things to be arranged (in this case $n = 15$ students) and how many will be taken (r) at a time. The formula for a permutation is $_{n}P_{r} = \frac{n!}{(n-r)!}$.

In part (a), we have 15 students taken 8 at a time.
The number of permutations is: $_{15}P_{8} = \frac{15!}{(15-8)!} = \frac{15!}{7!} = 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 259,459,200$

In part (b) the solution becomes: ${}_{15}P_2 = \frac{15!}{(15-2)!} = \frac{15!}{13!} = 15 \cdot 14 = 210$

Part (c) poses a new problem: ${}_{15}P_{15} = \frac{15!}{(15-15)!} = \frac{15!}{0!}$

What is 0! ? "Factorial" means to calculate the product of the integers from the given value down to one. How can we compute 0! ? If it equals zero, we have a problem because part (c) would not have an answer (dividing by zero is undefined). But this situation must have an answer. In fact, if we used a decision chart to determine how many ways the 15 people can line up, we would find that there are 15! arrangements. Therefore, if ${}_{15}P_{15} = 15!$ and $0! = 1$. This is another case of mathematicians defining elements of mathematics to fit their needs. 0! is defined to equal 1 so that other mathematics makes sense.

Example 3

In the annual homecoming parade, three students get to ride on the lead float. Seven students are being considered for this coveted position. How many ways can three students be chosen for this honor?

All three students who are selected will ride on the lead float, but whether they are the first, second, or third student selected does not matter. In a case where the order of the selections does not matter, the situation is called a **combination**. This means that if the students were labeled A, B, C, D, E, and F, choosing A, B, and then C would be essentially the same as choosing B, C, and then A. In fact, all the arrangements of A, B, and C could be lumped together. This makes the number of combinations much smaller than the number of permutations. The symbol for a combination is ${}_nC_r$ where n is the total number of items under consideration, and r is the number of items we will choose. It is often read as "*n* choose *r*." In this problem we have ${}_7C_3$, 7 choose 3. The formula is similar to the formula for a permutation, but we must divide out the similar groups.

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

Here we have: ${}_7C_3 = \frac{7!}{(7-3)!3!} = \frac{7!}{4!3!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1\cdot3\cdot2\cdot1} = 35$

Problems

Simplify the following expressions.

1. $10!$	2. $\frac{10!}{3!}$	3. $\frac{35!}{30!}$	4. $\frac{88!}{87!}$
5. $\frac{72!}{70!}$	6. $\frac{65!}{62!3!}$	7. ${}_8P_2$	8. ${}_{15}P_0$
9. ${}_9P_9$	10. ${}_{12}C_4$	11. ${}_5C_0$	12. ${}_{32}C_{32}$

Solve the following problems.

13. How many ways can you arrange the letters from the word "KAREN"?

14. How many ways can you arrange the letters from the word "KAREN" if you want the arrangement to begin with a vowel?

15. All standard license plates in Alaska start with three letters followed by three digits. If repetition is allowed, how many different license plates are there?

16. For $3.99, The Creamery Ice Cream Parlor will put any three different flavored scoops, out of their 25 flavors of ice cream, into a bowl. How many different "bowls" are there? (Note: A bowl of chocolate, strawberry, and vanilla is the same bowl as a bowl of chocolate, vanilla, and strawberry.)

17. Suppose those same three scoops of ice cream are on a cone. Now how many arrangements are there? (Note: Ice cream on a cone must be eaten "top down" because you cannot eat the bottom or middle scoop out, keeping the cone intact.)

18. A normal deck of playing cards contains 52 cards. How many five-card poker hands can be made?

19. How many ways are there to make a full house (three of one kind, two of another)?

20. What is the probability of getting a full house (three of one kind of card and two of another)? Assume a standard deck and no wild cards.

For problems 21–25, a bag contains 36 marbles. There are twelve blue marbles, eight red marbles, seven green marbles, five yellow marbles, and four white marbles. Without looking, you reach into the bag and pull out eight marbles. What is the probability you pull out:

21. All blue marbles?

22. Four blue and four white marbles?

23. Seven green and one yellow marble?

24. At least one red and at least two yellow?

25. No blue marbles?

Answers

1.	3,628,800	2.	604,800	3.	38,955,840	4.	88
5.	5,112	6.	43,680	7.	56	8.	1
9.	362,880	10.	495	11.	1	12.	1
13.	5! = 120	14.	2(4!) = 48	15.	(26)(26)(26)(10)(10)(10) = 17,576,000		

16. ${}_{25}C_3 = 2300$

17. ${}_{25}P_3 = 13{,}800$ (On a cone, order matters!)

18. ${}_{52}C_5 = 2{,}598{,}960$

19. This is tricky and tough! There are 13 different "types" of cards: twos, threes, fours, ..., Jacks, Queens, Kings, and Aces. We need to choose which of the 13 we want three of (${}_{13}C_1$). Once we choose what type (for example, we pick Jacks) then we need to choose which three out of the four to take (${}_{4}C_3$). Then from the remaining 12 types, we choose which type to have two of (${}_{12}C_1$). Then again we need to choose which two out of the four (${}_{4}C_2$). This gives us $({}_{13}C_1) \cdot ({}_{4}C_3) \cdot ({}_{12}C_1) \cdot ({}_{4}C_2) = 3744$.

20. We already calculated the numbers we need in problems 18 and 19 so: $\frac{3{,}744}{2{,}598{,}960} \approx 0.0014$.

21. Each time we reach in and pull out 8 marbles, order does not matter. The number of ways to do this is ${}_{36}C_8$. This is the number in the sample space, i.e., the denominator. How many ways can we pull out all blue? ${}_{12}C_8$. Therefore the probability is $\frac{{}_{12}C_8}{{}_{36}C_8} \approx 0.0000164$.

22. Same denominator. Now we want to choose 4 from the 12 blue, ${}_{12}C_4$, and 4 from the 4 whites, ${}_{4}C_4$. $\frac{{}_{12}C_4 \cdot {}_{4}C_4}{{}_{36}C_8} \approx 0.0000164$, the same answer!

23. Seven green: ${}_{7}C_7$, one yellow: ${}_{5}C_1$. $\frac{{}_{7}C_7 \cdot {}_{5}C_1}{{}_{36}C_8} \approx 0.0000001652$

24. Here we have to get at least one red: ${}_{8}C_1$, and at least two yellow: ${}_{5}C_2$, but the other five marbles can come from the rest of the pot: ${}_{33}C_5$. Therefore, $\frac{{}_{8}C_1 \cdot {}_{5}C_2 \cdot {}_{33}C_5}{{}_{36}C_8} \approx 0.627$.

25. To get no blue marbles means we want all eight from the other 24 non-blue marbles. $\frac{{}_{24}C_8}{{}_{36}C_8} \approx 0.0243$.

